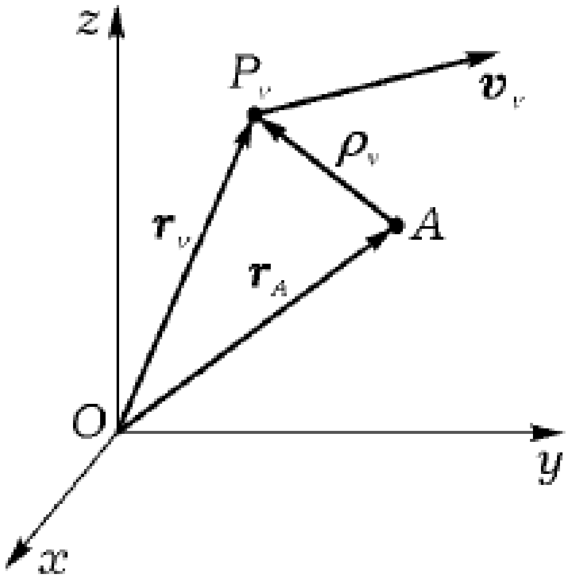
Attitude dynamics

1 – Main aspects of the rigid body dynamics

Doroshin A.V.

Samara State Aerospace University

The main dynamical parameters of rigid body motion about fixed point



The angular momentum:

$$\boldsymbol{K}_{\nu A} = \boldsymbol{\rho}_\nu \times m_\nu \boldsymbol{v}_\nu.$$

$$\boldsymbol{K}_A = \sum_{\nu=1}^{N} \boldsymbol{\rho}_\nu \times m_\nu \boldsymbol{v}_\nu.$$

The (linear) momentum | quantity of motion:

$$\boldsymbol{Q} = \sum_{\nu=1}^{N} m_\nu \boldsymbol{v}_\nu.$$

The kinetic energy: $T = \frac{1}{2} \sum_{\nu=1}^{N} m_\nu v_\nu^2.$

The angular momentum at the change of the point-pole (A to B):

$$\boldsymbol{K}_B = \boldsymbol{K}_A + \overline{BA} \times \boldsymbol{Q}.$$

The angular momentum and the kinetic energy of the rigid body with fixed point:

$$\boldsymbol{K}_O = \mathbf{J}\boldsymbol{\omega}.$$

$$T = \frac{1}{2}(\boldsymbol{K}_O \cdot \boldsymbol{\omega}).$$

$$K_{Ox} = J_x p - J_{xy} q - J_{xz} r,$$

$$K_{Oy} = -J_{xy} p + J_y q - J_{yz} r,$$

$$K_{Oz} = -J_{xz} p - J_{yz} q + J_z r.$$

$$T = \frac{1}{2}(J_x p^2 + J_y q^2 + J_z r^2) - J_{xy} pq - J_{xz} pr - J_{yz} qr.$$

$$T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2)$$

The dynamical theorems

The change of the angular momentum:

$$\frac{d\boldsymbol{K}_A}{dt} = M\boldsymbol{v}_C \times \boldsymbol{v}_A + \boldsymbol{M}_A^{(e)}.$$

the vector of **EXTERNAL** torques

The change of the kinetic energy:

$$dT = d'A^{(e)} + d'A^{(i)}.$$

the elementary mechanical work of all **EXTERNAL** and **INTERNAL** forces

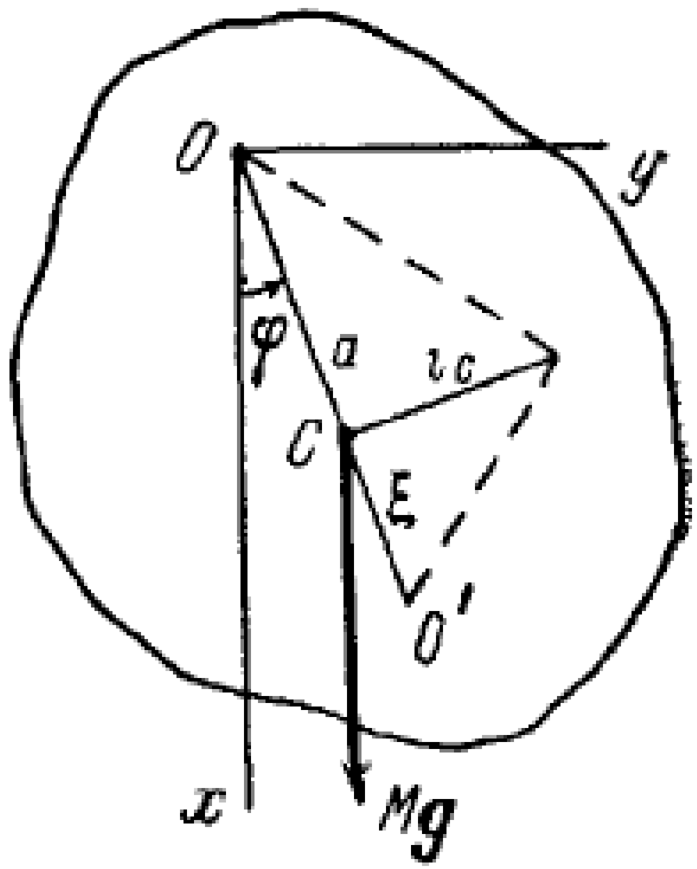
The potential (the partial case of forces):

$$d'A^{(e)} + d'A^{(i)} = -d\Pi.$$

$$dT + d\Pi = 0.$$

$$E = T + \Pi = h = \text{ const}$$

The physical pendulum angular motion as an important example of rigid body motion



Point A → fixed point O

$$\frac{d\mathbf{K}_A}{dt} = M\mathbf{v}_C \times \mathbf{v}_A + \mathbf{M}_A^{(e)}.$$

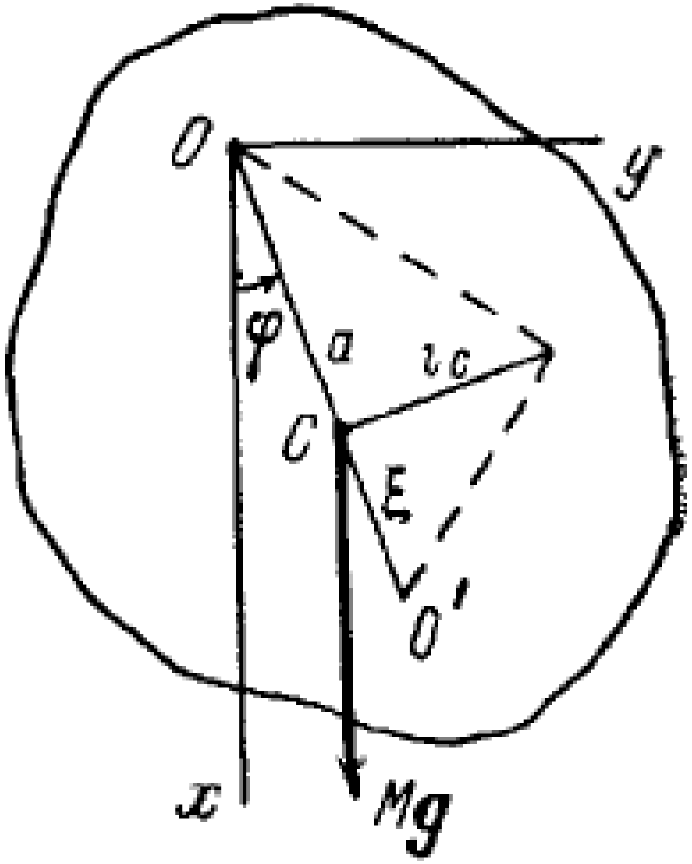
$M\mathbf{v}_C \times \mathbf{v}_A$ → 0

$$J_O \frac{d^2\varphi}{dt^2} = -Mga\sin\varphi,$$

$$\frac{d^2\varphi}{dt^2} + \frac{Mga\sin\varphi}{J_O} = 0.$$

$$\frac{d^2\varphi}{dt^2} + \frac{g}{l}\sin\varphi = 0, \qquad l = \frac{J_O}{Ma},$$

The physical pendulum angular motion as an important example of rigid body motion



$$T = \frac{1}{2} J_z \dot{\varphi}^2, \qquad \Pi = -mga\cos\varphi.$$

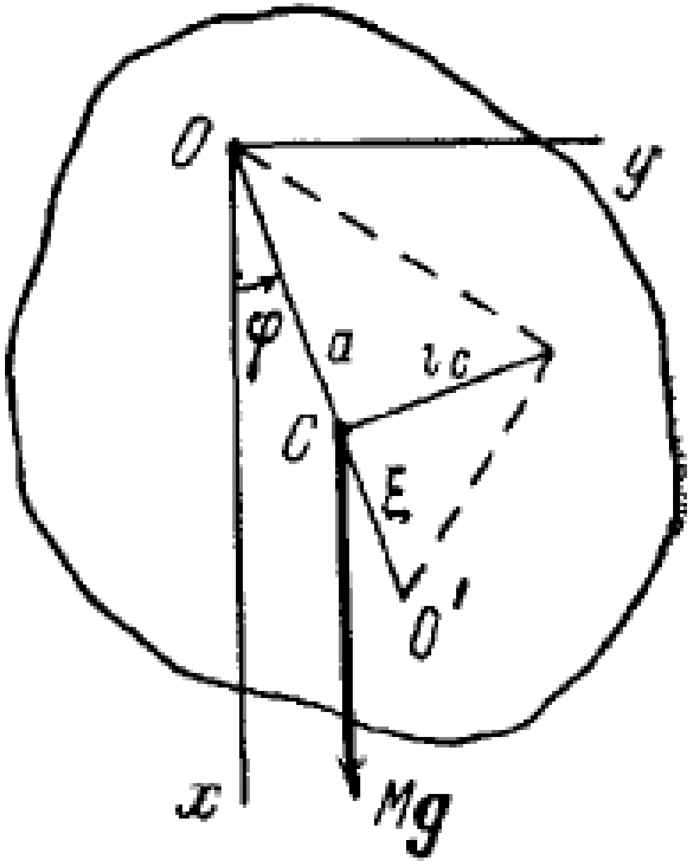
$$\omega_0^2 = g/l, \;\; \Pi^* = -\omega_0^2 \cos\varphi,$$

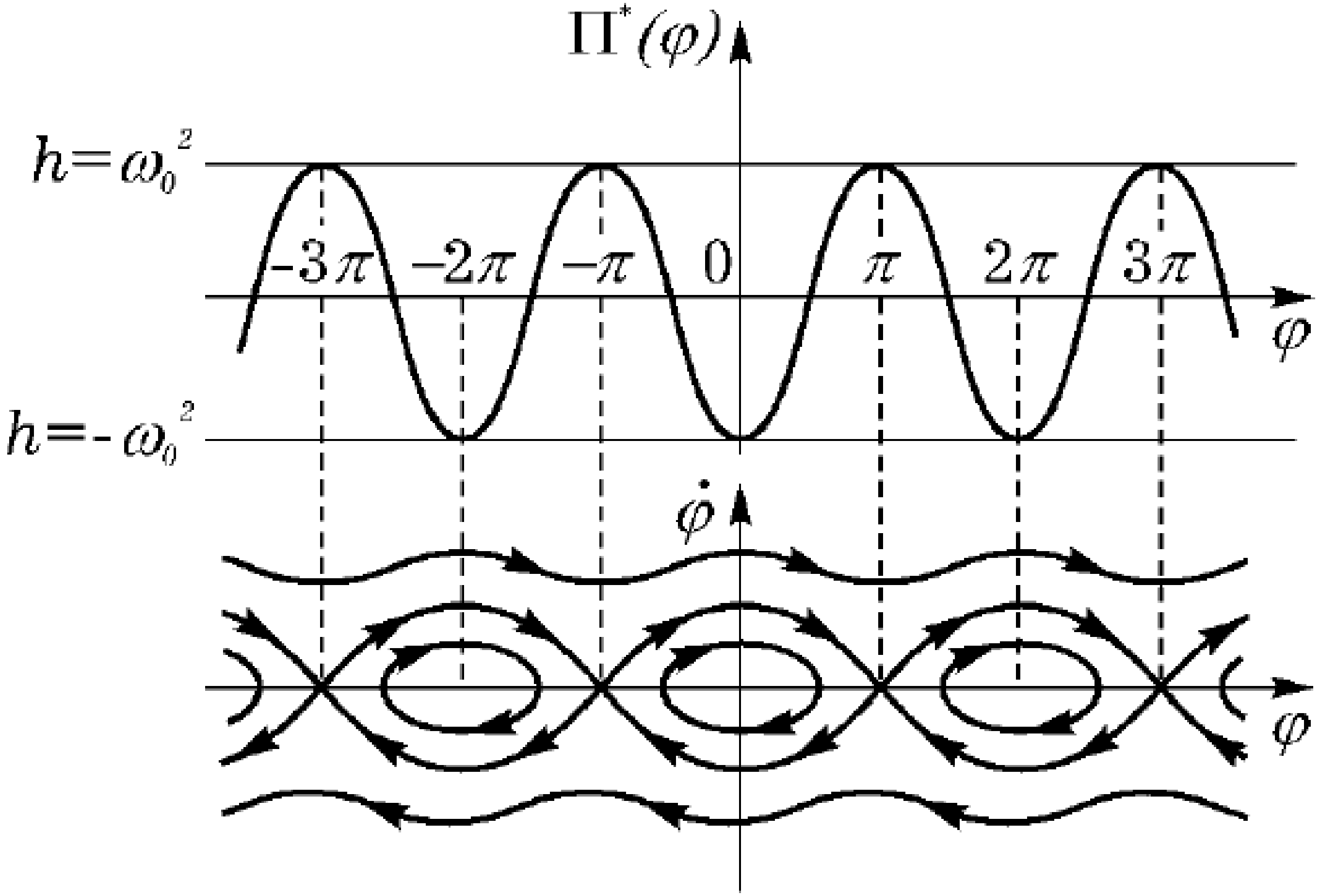
$$T + \Pi = \text{const}$$

$$\frac{1}{2}\dot{\varphi}^2 + \Pi^* = h = \text{const}.$$

The physical pendulum angular motion as an important example of rigid body motion

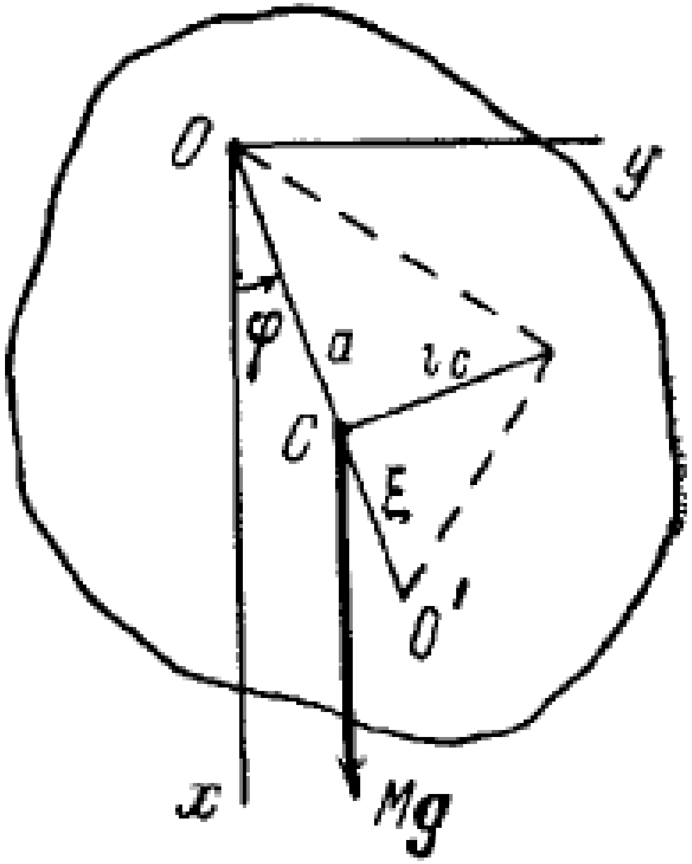
$$\frac{1}{2}\dot{\varphi}^2 + \Pi^* = h = \text{ const.}$$





The physical pendulum angular motion as an important example of rigid body motion

Some aspects of elliptic functions



$$u = F(\varphi,\ k) = \int\limits_0^{\varphi} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

$$0 \leqslant k < 1.$$

The Jacobi elliptical integral (the first kind elliptic integral)

$$K(k) = F\left(\frac{\pi}{2},\ k\right) = \int\limits_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

Then:

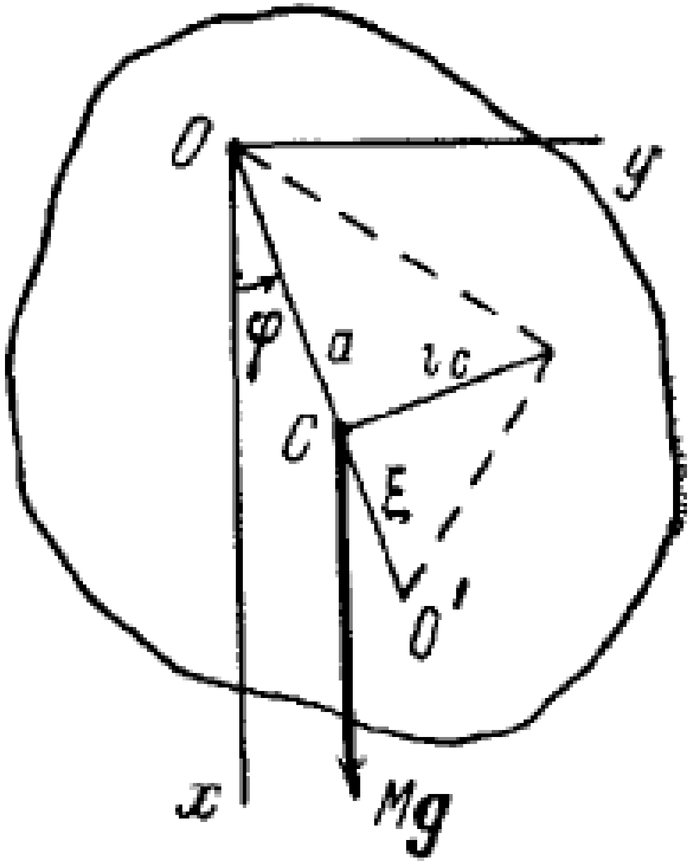
$$\varphi = \operatorname{am} u.$$

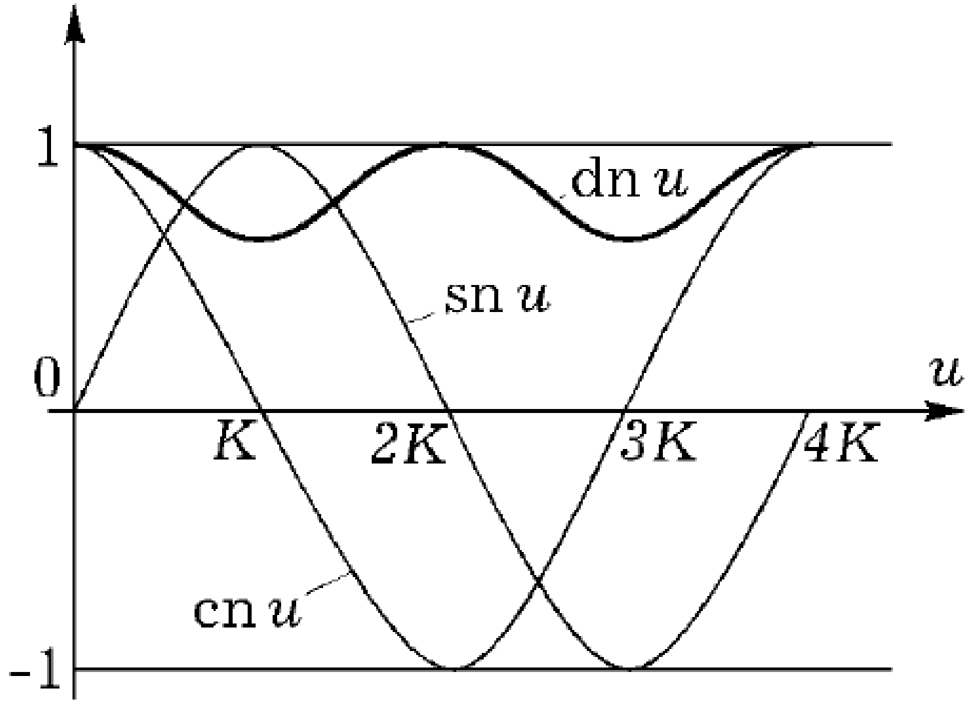
$$z = \operatorname{sn}(u,\ k) = \sin\varphi = \sin \operatorname{am} u \quad \text{и} \quad z = \operatorname{cn}(u,\ k) = \cos\varphi = \cos \operatorname{am} u.$$

$$z = \operatorname{dn}(u,\ k) = \frac{d\varphi}{du} = \sqrt{1 - k^2 \sin^2 \varphi} = \sqrt{1 - k^2 \operatorname{sn}^2(u,\ k)}.$$

The physical pendulum angular motion as an important example of rigid body motion

Some aspects of elliptic functions

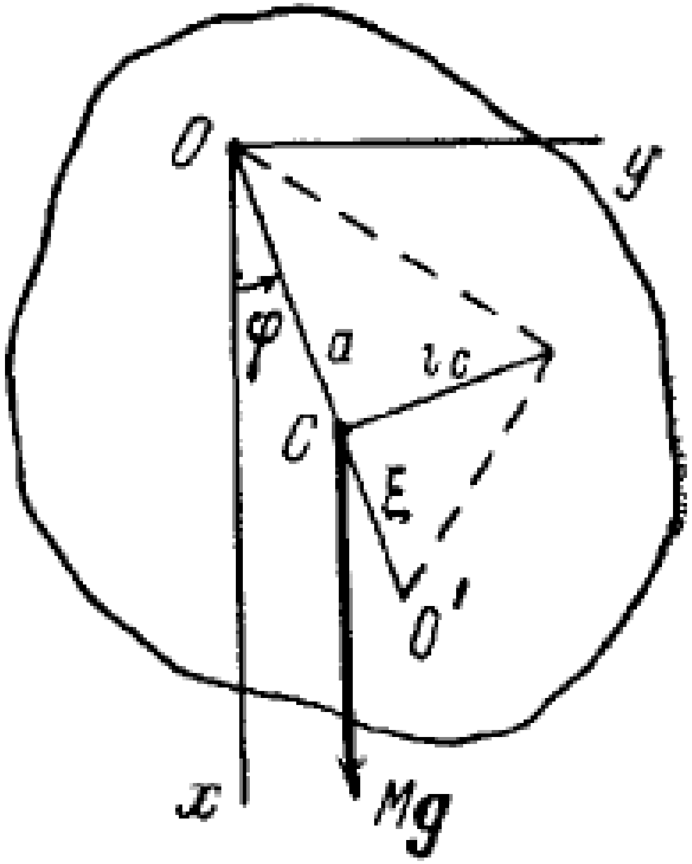




$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1,$$
$$\operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u = 1.$$

$$\frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \cdot \operatorname{dn} u,$$
$$\frac{d}{du} \operatorname{cn} u = - \operatorname{sn} u \cdot \operatorname{dn} u,$$
$$\frac{d}{du} \operatorname{dn} u = -k^2 \operatorname{sn} u \cdot \operatorname{cn} u.$$

The physical pendulum angular motion as an important example of rigid body motion



$$T = \frac{1}{2} J_z \dot{\varphi}^2, \qquad \Pi = -mga\cos\varphi.$$

$$\omega_0^2 = g/l, \quad \Pi^* = -\omega_0^2 \cos\varphi,$$

$$\frac{1}{2}\dot{\varphi}^2 + \Pi^* = h = \text{const.}$$

$$h = -\omega_0^2 \cos\beta$$

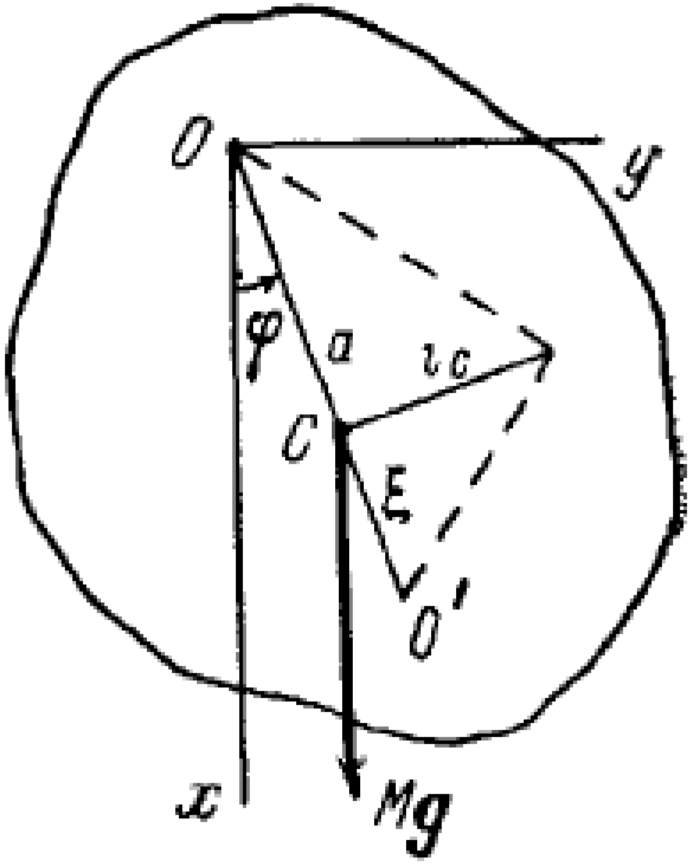
β – the maximum of φ value

$$\dot{\varphi}^2 = 2\omega_0^2(\cos\varphi - \cos\beta).$$

$$-\omega_0^2 < h < \omega_0^2.$$

Changing variables $\sin(\varphi/2) = k_1 \sin\psi.$ → $\dot{\psi}^2 = \omega_0^2(1 - k_1^2 \sin^2\psi).$ $k_1 = \sin(\beta/2)$

The physical pendulum angular motion as an important example of rigid body motion



$$\dot{\psi}^2 = \omega_0^2(1 - k_1^2 \sin^2 \psi).$$

$$t = 0 \quad \varphi = 0,$$

$$\omega_0 t = \int_0^{\psi} \frac{dx}{\sqrt{1 - k_1^2 \sin^2 x}} = F(\psi,\ k_1),$$

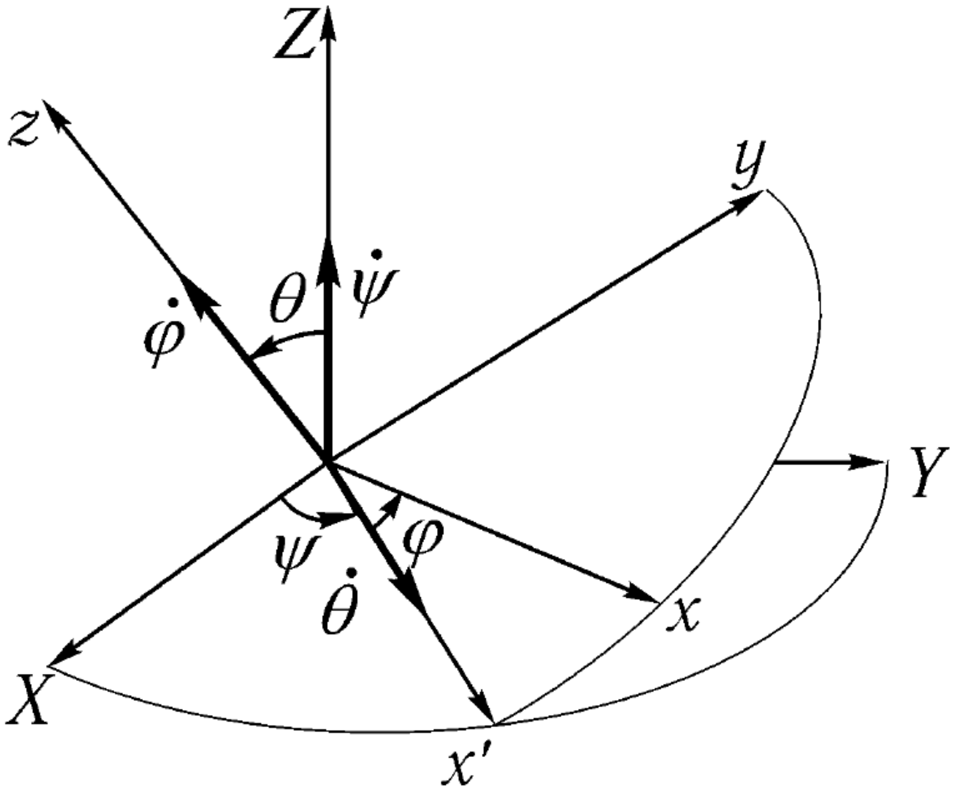
$$\psi = \mathrm{am}(\omega_0 t).$$

$$\sin(\varphi/2) = k_1 \sin \psi.$$

$$\varphi = 2 \arcsin(k_1 \,\mathrm{sn}\, \omega_0 t).$$

If the angle is small then we can linearize the equation; and as the result t we obtain the harmonic solutions with ordinal SINUS → the practical seminar

Equations of the angular motion of rigid body about the fixed point



Rotations:
the precession ψ-> the nutation θ -> the intrinsic rotation φ

$$\frac{d\boldsymbol{K}_O}{dt} = \boldsymbol{M}_O^{(e)}.$$

$$A\dot{p} + (C - B)qr = M_x,$$
$$B\dot{q} + (A - C)rp = M_y,$$
$$C\dot{r} + (B - A)pq = M_z.$$

$$p = \dot\psi \sin\theta \sin\varphi + \dot\theta \cos\varphi,$$
$$q = \dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi,$$
$$r = \dot\psi \cos\theta + \dot\varphi$$

Equations of the angular motion in the ***Euler case***

Leonhard Euler

Born: April 15, 1707,
Basel, Switzerland

Died: September 18, 1783,
Saint Petersburg, Russian Empire

$$\frac{d\boldsymbol{K}_O}{dt} = \boldsymbol{M}_O^{(e)}. \quad (\boldsymbol{M}_O^{(e)} \to 0)$$

$$\boldsymbol{K}_O = \text{const},$$

$$K_O^2 = A^2p^2 + B^2q^2 + C^2r^2 = \text{const}.$$

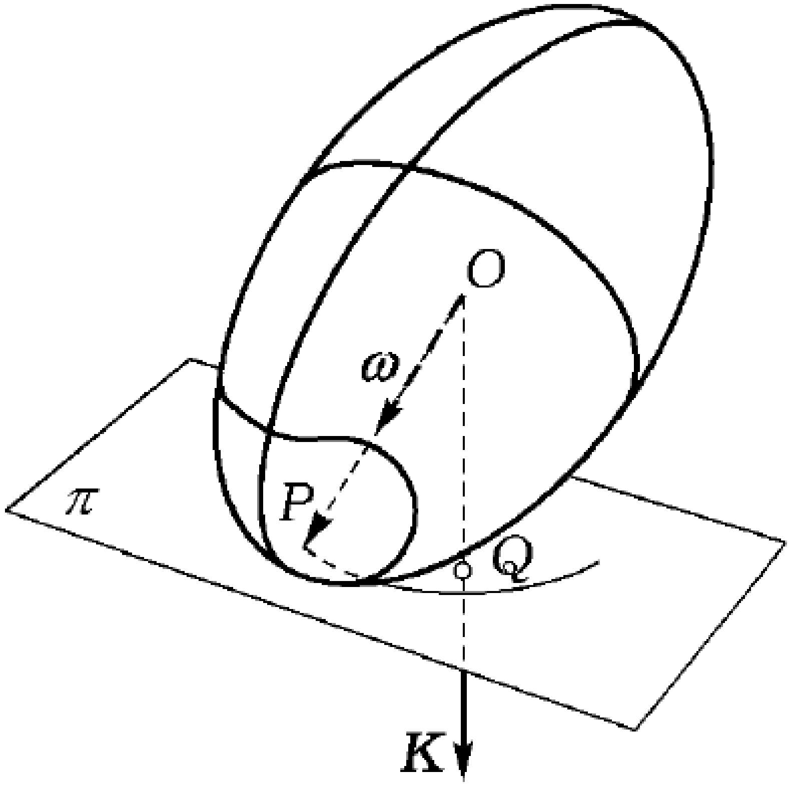
$$dT = 0.$$

$$T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) = \text{const}.$$

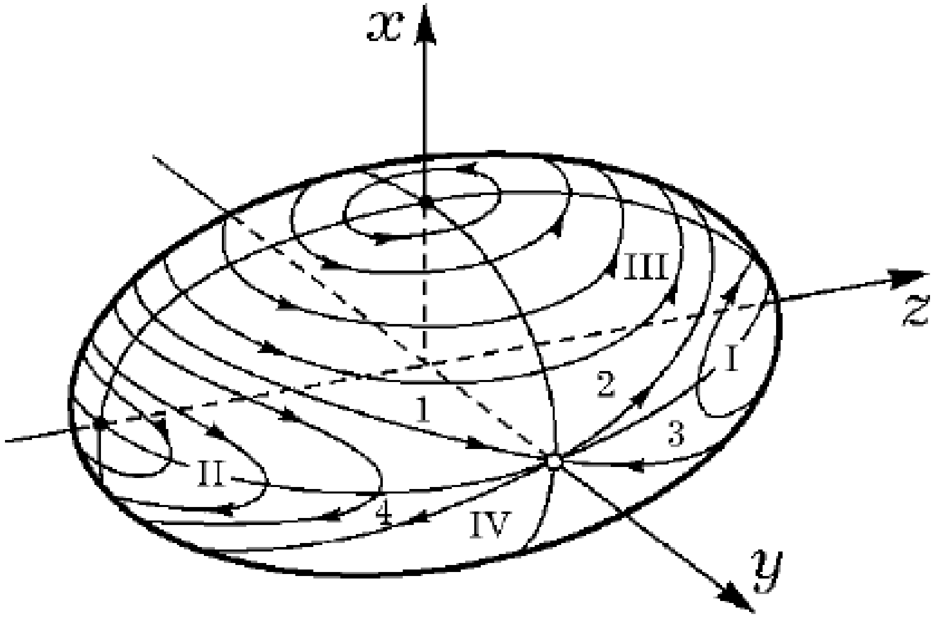
$$A\dot{p} + (C - B)qr = 0,$$
$$B\dot{q} + (A - C)rp = 0,$$
$$C\dot{r} + (B - A)pq = 0.$$

Equations of the angular motion in the Euler case

Poinsot's construction. Polhodes

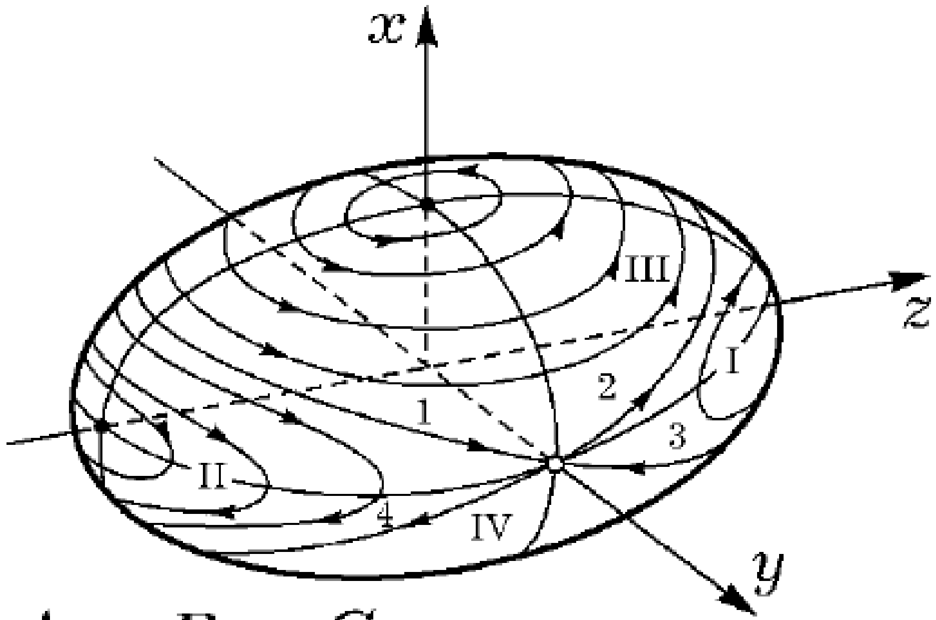


$$OQ = \frac{\boldsymbol{K}_O \cdot \overline{OP}}{K_O} = \lambda \frac{\boldsymbol{K}_O \cdot \boldsymbol{\omega}}{K_O} = \lambda \frac{2T}{K_O} = \frac{\sqrt{2T}}{K_O} = \text{const.}$$



https://www.youtube.com/watch?v=BwYFT3T5uIw

Equations of the angular motion in the Euler case



$$A > B > C.$$

$$K_O^2 = A^2p^2 + B^2q^2 + C^2r^2 = \text{const.}$$

$$T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) = \text{const.}$$

$$p^2 = \frac{1}{A(C-A)}\left[(2TC - K_O^2) - B(C-B)q^2\right],$$

$$r^2 = \frac{1}{C(C-A)}\left[(K_O^2 - 2TA) - B(B-A)q^2\right].$$

$$B\dot{q} + (A-C)rp = 0,$$

$$\frac{dq}{dt} = \pm\frac{1}{B\sqrt{AC}}\sqrt{[(2TC - K_O^2) - B(C-B)q^2][(K_O^2 - 2TA) - B(B-A)q^2]}.$$

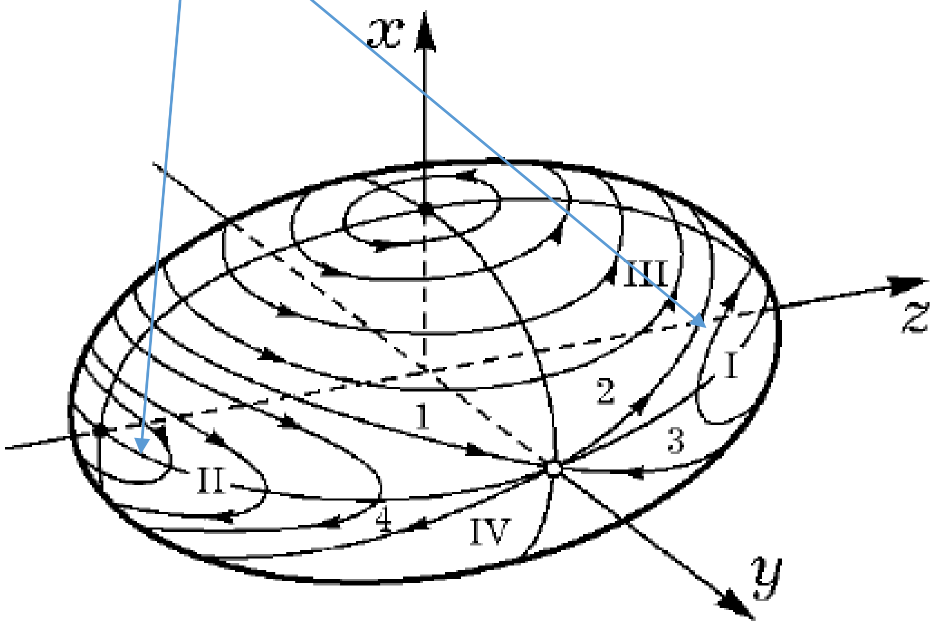
Equations of the angular motion in the Euler case

$$\frac{dq}{dt} = \pm\frac{1}{B\sqrt{AC}}\sqrt{[(2TC - K_O^2) - B(C-B)q^2][(K_O^2 - 2TA) - B(B-A)q^2]}.$$

Case 1

$$2TB > K_O^2 \geqslant 2TC.$$

Areas I and II



Changing variables

$$q = \pm\sqrt{\frac{K_O^2 - 2TC}{B(B-C)}}\sin\lambda,$$

$$\tau = \sqrt{\frac{(B-C)(2TA - K_O^2)}{ABC}}t$$

$$\frac{d\lambda}{d\tau} = \sqrt{1 - k^2\sin^2\lambda}.$$

$$k^2 = \frac{(A-B)(K_O^2 - 2TC)}{(B-C)(2TA - K_O^2)}.$$

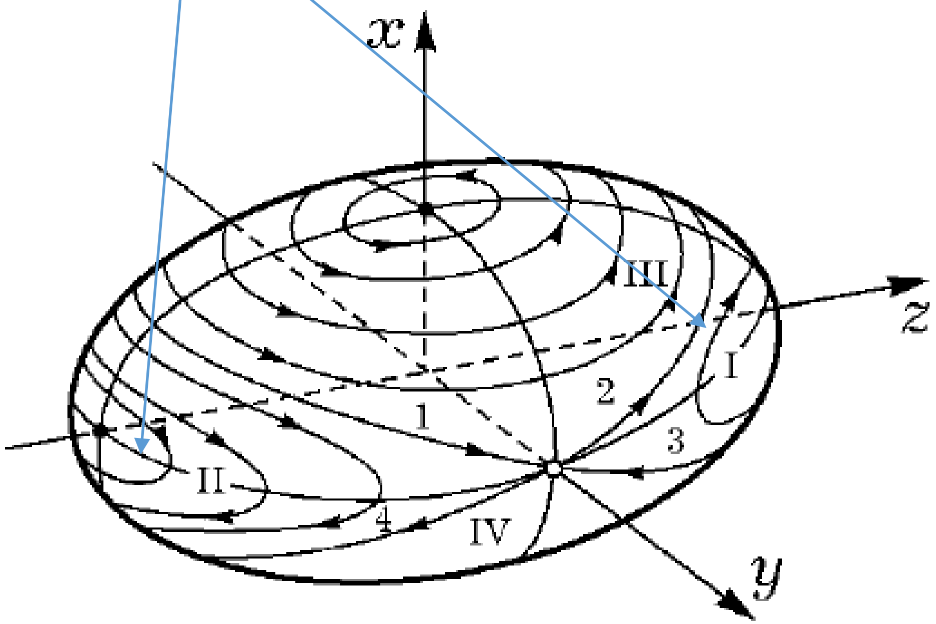
$$\lambda = \operatorname{am}\tau.$$

Equations of the angular motion in the Euler case

$$t = 0 \; q = 0.$$

$$2TB > K_O^2 \geqslant 2TC.$$

Areas I and II



$$\lambda = \operatorname{am} \tau.$$

By back substitutions

$$p = \mp\sqrt{\frac{K_O^2 - 2TC}{A(A-C)}}\,\operatorname{cn}(\tau,\ k),$$

$$q = \pm\sqrt{\frac{K_O^2 - 2TC}{B(B-C)}}\,\operatorname{sn}(\tau,\ k)$$

$$r = \sqrt{\frac{2TA - K_O^2}{C(A-C)}}\,\operatorname{dn}(\tau,\ k).$$

Equations of the angular motion in the Euler case

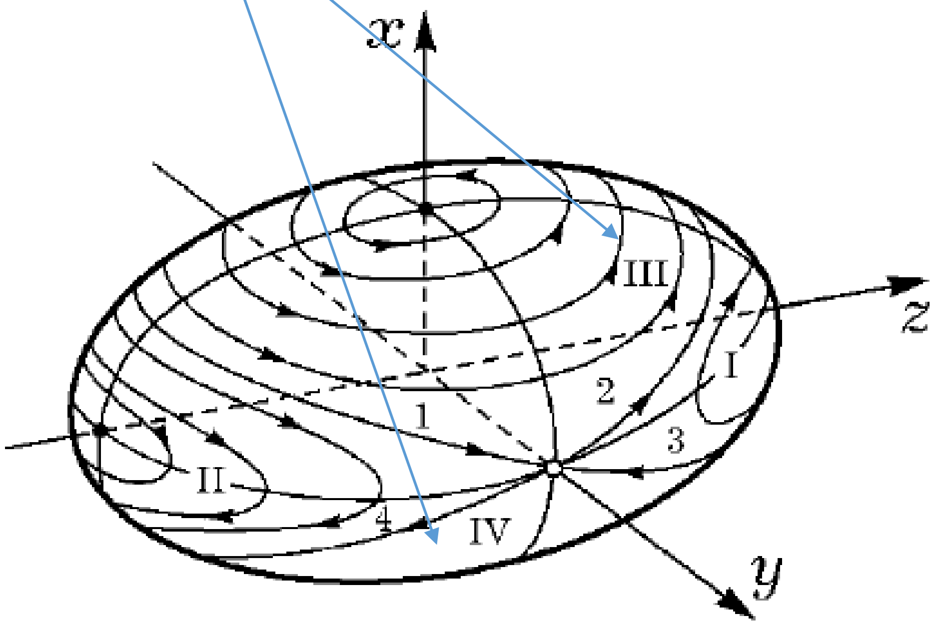
Case 2

$$2TA \geqslant K_O^2 > 2TB.$$

Changing of variables

$$q = \pm\sqrt{\frac{2TA - K_O^2}{B(A-B)}}\sin\lambda, \qquad \tau = \sqrt{\frac{(A-B)(K_O^2 - 2TC)}{ABC}}\,t.$$

Areas III and IV



$$k^2 = \frac{(B-C)(2TA - K_O^2)}{(A-B)(K_O^2 - 2TC)},$$

$$p = \sqrt{\frac{K_O^2 - 2TC}{A(A-C)}}\,\mathrm{dn}(\tau,\ k), \qquad q = \pm\sqrt{\frac{2TA - K_O^2}{B(A-B)}}\,\mathrm{sn}(\tau,\ k),$$

$$r = \mp\sqrt{\frac{2TA - K_O^2}{C(A-C)}}\,\mathrm{cn}(\tau,\ k).$$

Equations of the angular motion in the Euler case

Case 3

$$p^2 = \frac{(B-C)}{A(A-C)}(2T - Bq^2), \quad r^2 = \frac{(A-B)}{C(A-C)}(2T - Bq^2).$$

$$K_O^2 = 2TB.$$

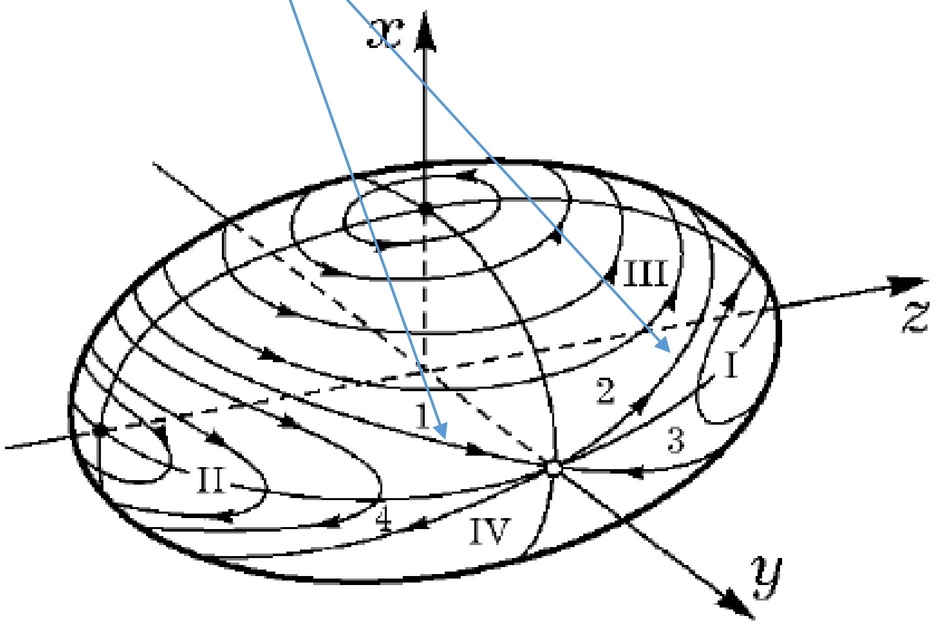
$$\tau = \sqrt{\frac{2T(A-B)(B-C)}{ABC}}\,t,$$

$$\frac{dq}{d\tau} = \pm\frac{1}{\sqrt{2TB}}(2T - Bq^2).$$

$$p = \sqrt{\frac{2T(B-C)}{A(A-C)}}\,\frac{1}{\operatorname{ch}\tau},$$

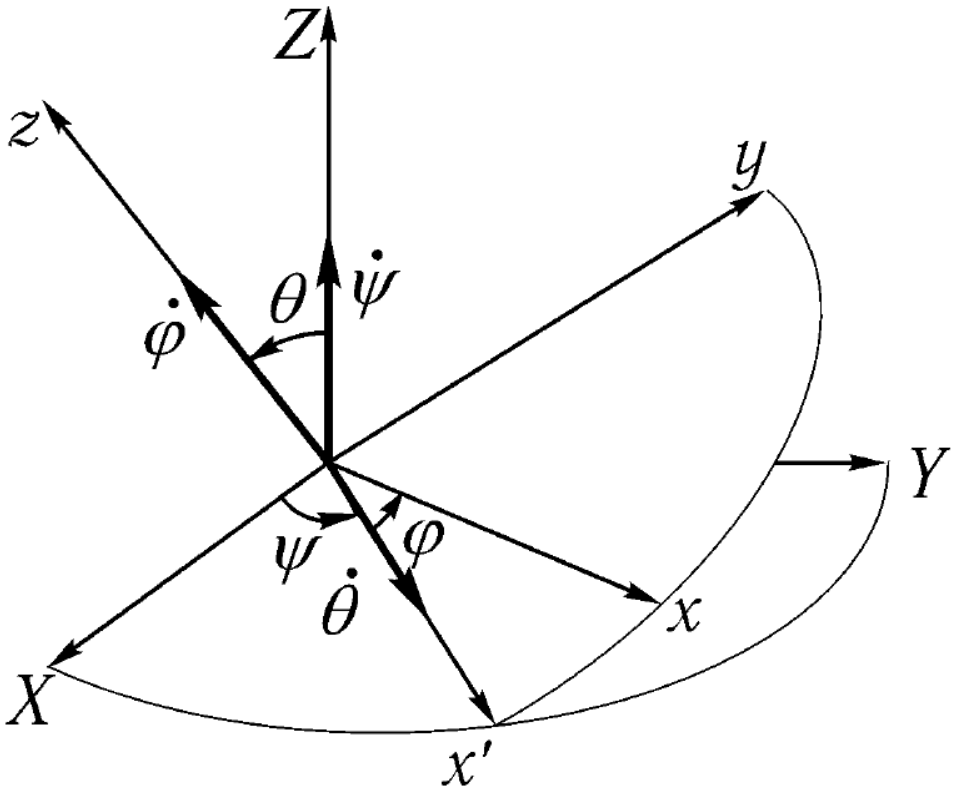
$$q = \sqrt{\frac{2T}{B}}\operatorname{th}\tau, \quad r = -\sqrt{\frac{2T(A-B)}{C(A-C)}}\,\frac{1}{\operatorname{ch}\tau}.$$

Trajectories 1 and 2



Equations of the angular motion in the Euler case

Solutions for Euler angles



Let us to direct the axes
with fulfilment of the condition

OZ ≡ K

$$Ap = K_O \sin\theta \sin\varphi,$$

$$Bq = K_O \sin\theta \cos\varphi, \quad Cr = K_O \cos\theta.$$

$$\cos\theta = \frac{Cr}{K_O}, \qquad \operatorname{tg}\varphi = \frac{Ap}{Bq}.$$

$$\dot\psi = \frac{p\sin\varphi + q\cos\varphi}{\sin\theta}. \rightarrow \dot\psi = \frac{Ap^2 + Bq^2}{K_O \sin^2\theta}. \rightarrow \dot\psi = K_O \frac{Ap^2 + Bq^2}{A^2p^2 + B^2q^2}. \rightarrow$$

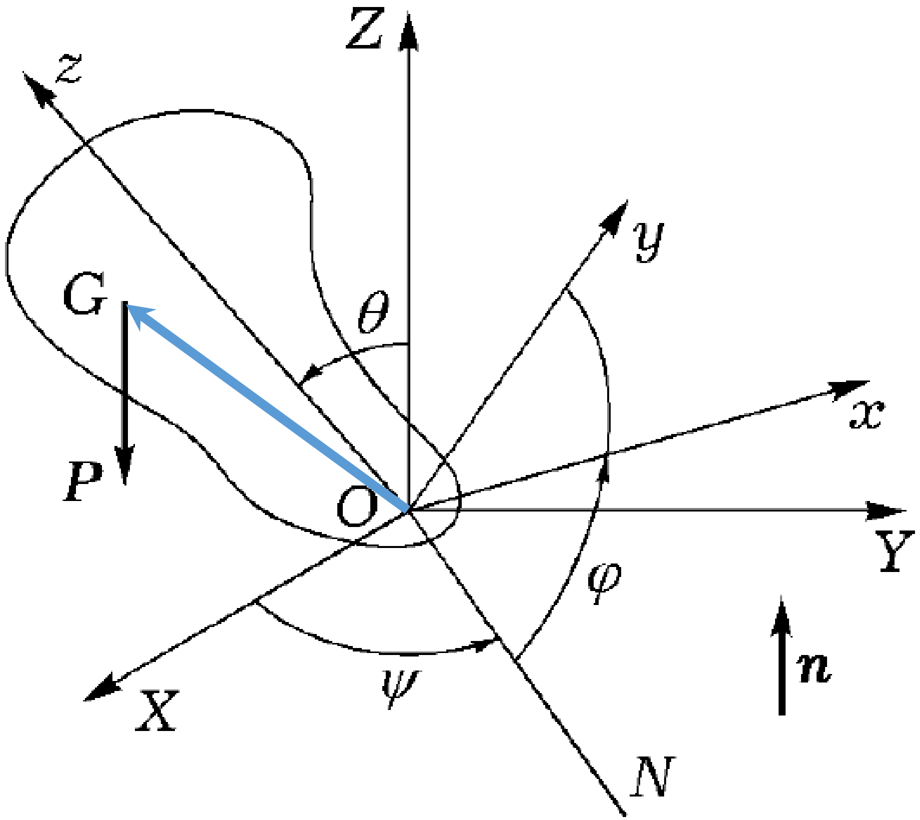
Integrating elliptic function…

Equations of the angular motion in the *Lagrange case*

Joseph-Louis Lagrange

Born: 25 January 1736,
Turin, Piedmont-Sardinia

Died: 10 April 1813 (aged 77)
Paris, France



$$\gamma_1 = \sin\theta\sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi,$$

$$\gamma_3 = \cos\theta.$$

$$\frac{d\mathbf{K}_O}{dt} = \mathbf{M}_O^{(e)}.$$

$$\boldsymbol{M}_O^{(e)} = P\boldsymbol{n} \times \overline{OG}.$$

$$M_x = P(\gamma_2 c - \gamma_3 b),$$

$$M_y = P(\gamma_3 a - \gamma_1 c),$$

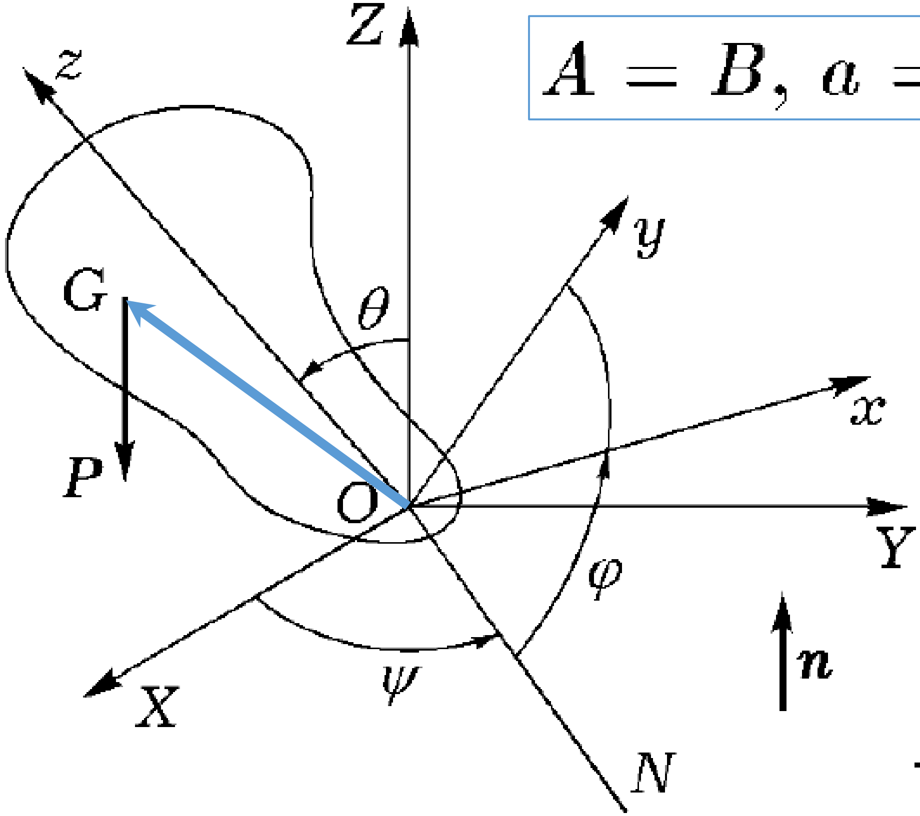
$$M_z = P(\gamma_1 b - \gamma_2 a).$$

$$d\boldsymbol{n}/dt = 0.$$

$$\frac{\tilde{d}\boldsymbol{n}}{dt} + \boldsymbol{\omega} \times \boldsymbol{n} = 0,$$

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3, \quad \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1, \quad \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2.$$

Equations of the angular motion in the ***Lagrange case***



$$A = B, \; a = b = 0.$$

$$A\frac{dp}{dt} + (C - B)qr = P(\gamma_2 c - \cancel{\gamma_3 b}),$$
$$B\frac{dq}{dt} + (A - C)rp = P(\cancel{\gamma_3 a} - \gamma_1 c),$$
$$C\frac{dr}{dt} + (B - A)pq = P(\cancel{\gamma_1 b} - \cancel{\gamma_2 a}).$$

$$\gamma_1 = \sin\theta \sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi,$$
$$\gamma_3 = \cos\theta.$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

<u>The first integrals:</u>

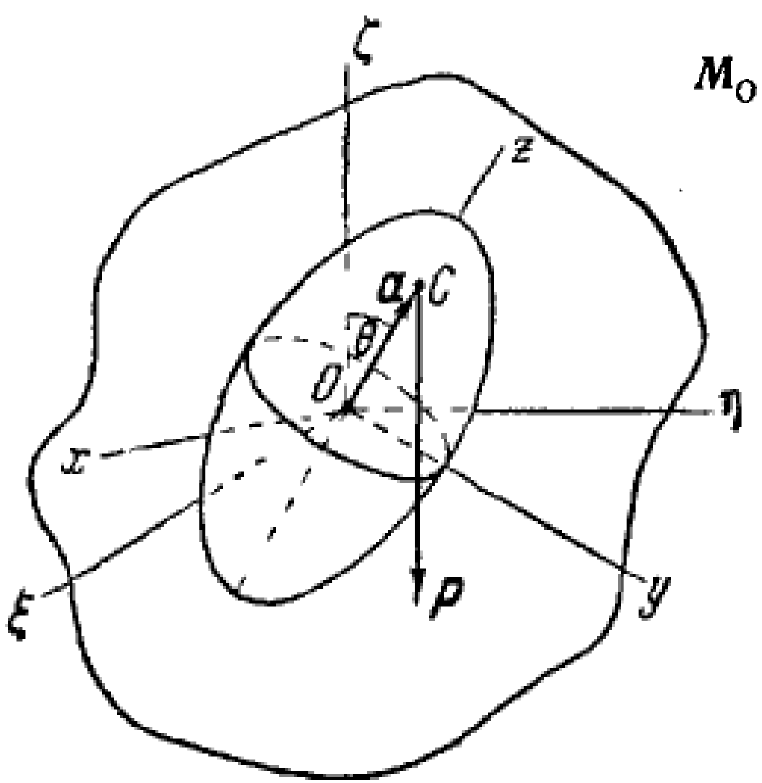
$$\boldsymbol{K}_O \cdot \boldsymbol{n} = \text{const.} \quad Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = \text{const.}$$

$$\Pi = Ph, \qquad h = \overline{OG} \cdot \boldsymbol{n} = \cancel{a}\gamma_1 + \cancel{b}\gamma_2 + c\gamma_3.$$

$$T = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2), \qquad E = T + \Pi$$

$$\frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + P(\cancel{a}\gamma_1 + \cancel{b}\gamma_2 + c\gamma_3) = \text{const.}$$

Equations of the angular motion in the ***Lagrange case***



In the "canonical" designation

$$M_O = \text{mom}_O\, \boldsymbol{P} = \boldsymbol{a} \times \boldsymbol{P} = P \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \gamma_1 & \gamma_2 & \gamma_3 \\ 0 & 0 & a \end{vmatrix},$$

$$M_x = Pa\gamma_2,\quad M_y = -Pa\gamma_1,\quad M_z = 0.$$

$$A\frac{dp}{dt} + (C - B)\, qr = Pa\gamma_2,$$
$$B\frac{dq}{dt} + (A - C)\, rp = -Pa\gamma_1,$$
$$C\frac{dr}{dt} = 0.$$

$$p = \dot\psi \sin\theta \sin\varphi + \dot\theta \cos\varphi,$$
$$q = \dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi,$$
$$r = \dot\varphi + \dot\psi \cos\theta.$$

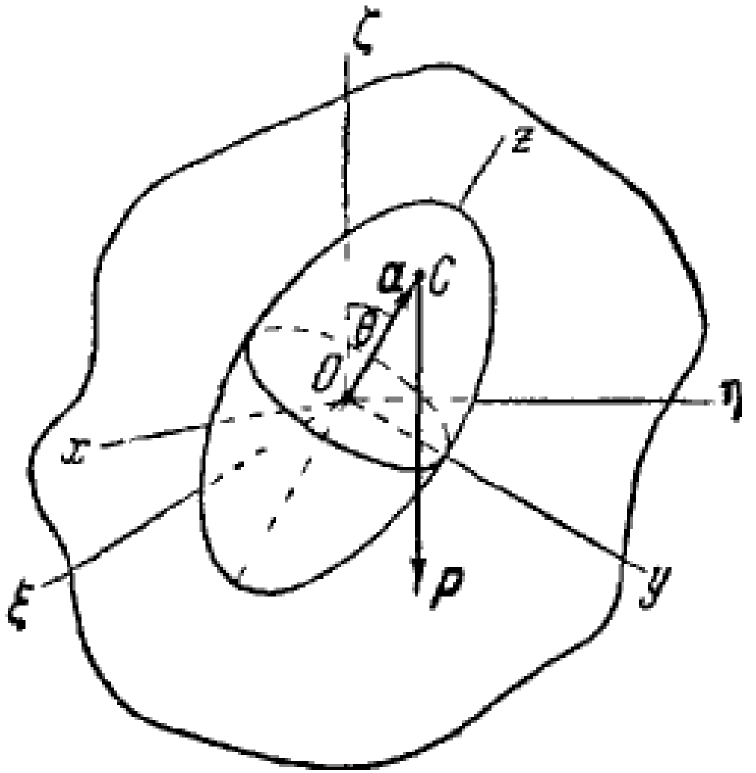
First integrals:

$$A(p^2 + q^2) + Cr^2 = -2Pa\gamma_3 + h;$$ - The energy conservation

$$A(p\gamma_1 + q\gamma_2) + Cr\gamma_3 = \text{const}.$$ - The "vertical" component of ***K*** conservation

$$r = \text{const}.$$ - The "logitudinal" component of ***K*** conservation

Equations of the angular motion in the *Lagrange case*



$$p = \dot\psi \sin\theta \sin\varphi + \dot\theta \cos\varphi,$$
$$q = \dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi,$$
$$r = \dot\varphi + \dot\psi \cos\theta.$$

1

$$p^2 + q^2 = \dot\psi^2 \sin^2\theta + \dot\theta^2,$$

The energy conservation, and $\gamma_3 = \cos\theta.$

$$A(\dot\psi^2 \sin^2\theta + \dot\theta^2) + 2Pa\cos\theta = h_1, \quad h_1 = h - Cr^2.$$

2

$$p\gamma_1 = \dot\psi \sin^2\theta \sin^2\varphi + \dot\theta \cos\varphi \sin\theta \sin\varphi,$$
$$q\gamma_2 = \dot\psi \sin^2\theta \cos^2\varphi - \dot\theta \sin\varphi \sin\theta \cos\varphi.$$

$$p\gamma_1 + q\gamma_2 = \dot\psi \sin^2\theta.$$

The “vertical” component of ***K*** conservation

$$A\dot\psi \sin^2\theta + Cr\cos\theta = b.$$

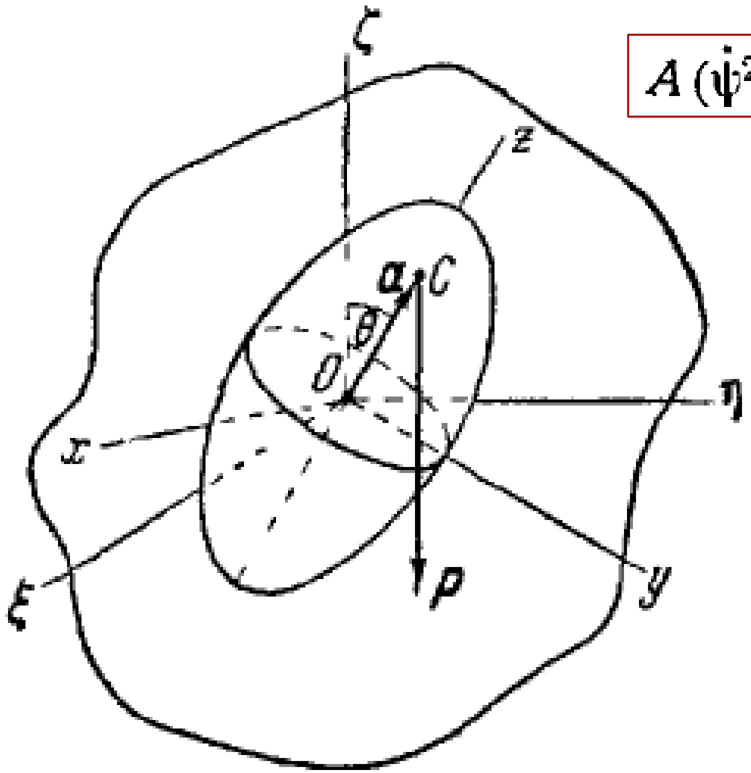
b - const

3

$$\dot\psi \cos\theta + \dot\varphi = r = \text{const.}$$

- The “logitudinal” component of ***K*** conservation

Equations of the angular motion in the ***Lagrange case***



$$A\dot{\psi}\sin^2\theta + Cr\cos\theta = b. \quad\Rightarrow\quad \dot{\psi} = \frac{b - Cr\cos\theta}{A\sin^2\theta}$$

$$A(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2) + 2Pa\cos\theta = h_1, \quad\Rightarrow\quad \dot{\theta}^2 = \frac{h_1 - 2Pa\cos\theta}{A} - \dot{\psi}^2\sin^2\theta;$$

$$\dot{\theta}^2 = \frac{h_1 - 2Pa\cos\theta}{A} - \frac{(b - Cr\cos\theta)^2}{A^2\sin^2\theta}.$$

By reducing:

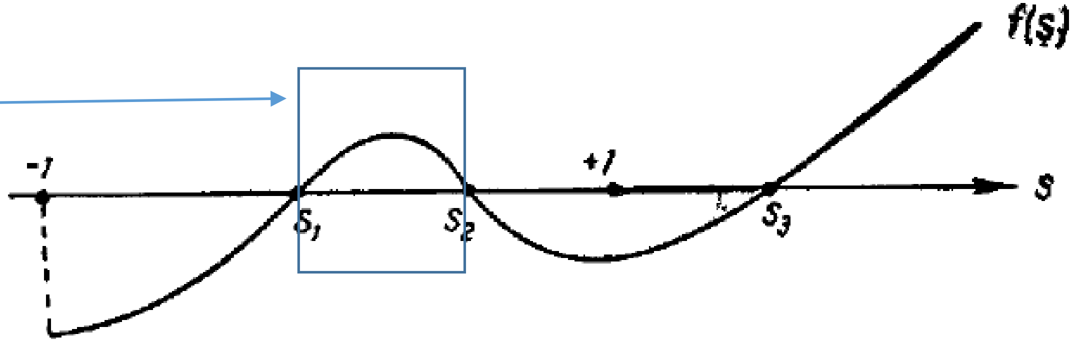
$$A^2\sin^2\theta\cdot\dot{\theta}^2 = A(h_1 - 2Pa\cos\theta)\sin^2\theta - (b - Cr\cos\theta)^2$$

The change of variables: $\cos\theta = s, \quad -\sin\theta\cdot\dot{\theta} = \frac{ds}{dt}.$

$$A^2\left(\frac{ds}{dt}\right)^2 = f(s), \quad f(s) = A(h_1 - 2Pas)(1 - s^2) - (b - Crs)^2.$$

The polynomial of the fourth power has the 3 real roots

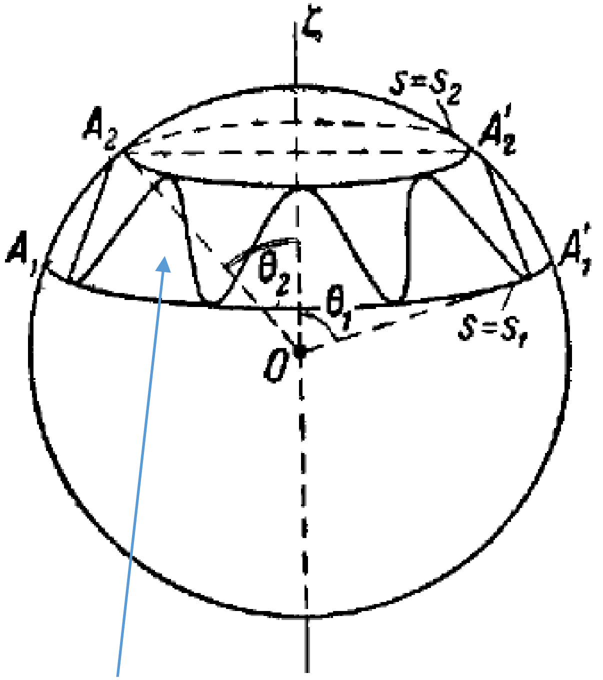
A real motion area



$$s_1 \leqslant s \leqslant s_2,$$

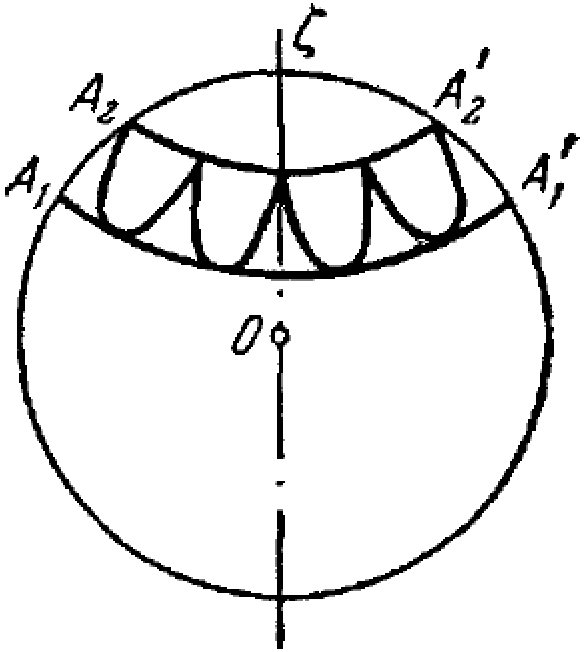
$$\cos\theta_1 \leqslant \cos\theta \leqslant \cos\theta_2,$$

Equations of the angular motion in the ***Lagrange case***

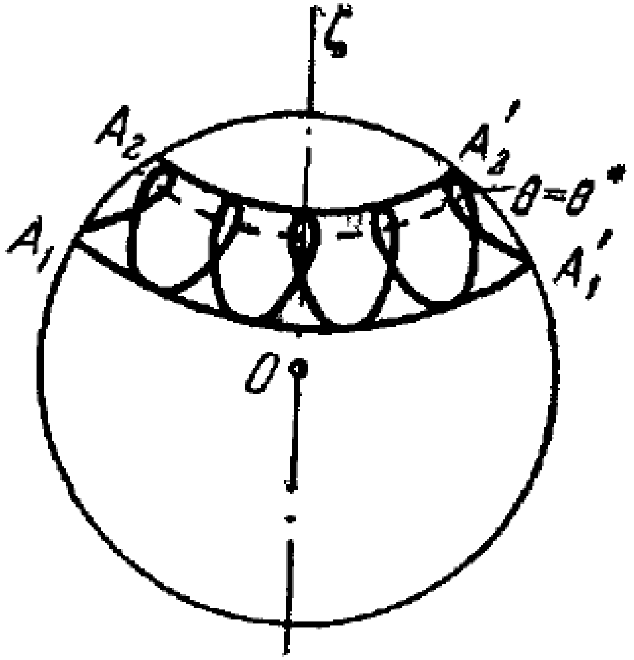


$b > Cr\cos\theta_2$

$b < Cr\cos\theta_1)$

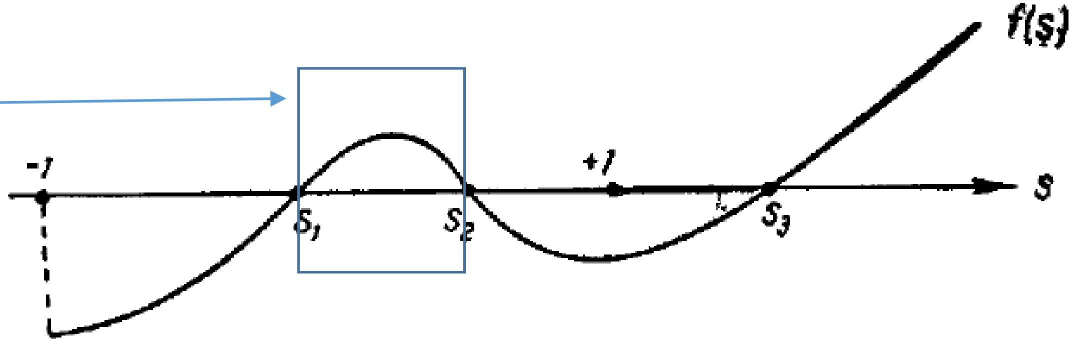


$b = Cr\cos\theta_2.$



$Cr\cos\theta_1 < b < Cr\cos\theta_2.$

A real motion area



$s_1 \leqslant s \leqslant s_2,$

$\cos\theta_1 \leqslant \cos\theta \leqslant \cos\theta_2,$

Equations of the angular motion in the *Lagrange case*

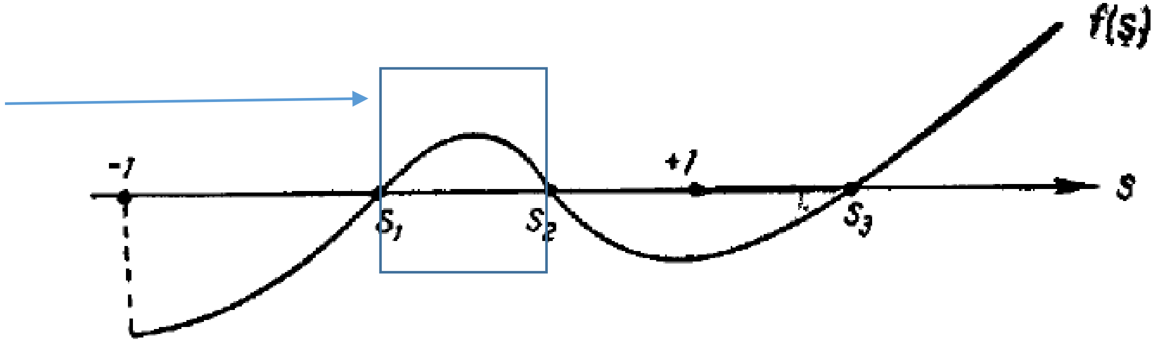
$$A^2\left(\frac{ds}{dt}\right)^2 = f(s), \quad \Longrightarrow \quad A\dot{s} = \pm\sqrt{f(s)}$$

Integrating in elliptic integrals:

$$t_{12} = A\int_{s_1}^{s_2}\frac{ds}{\sqrt{f(s)}}; \qquad t_{21} = A\int_{s_2}^{s_1}\frac{ds}{-\sqrt{f(s)}} = A\int_{s_1}^{s_2}\frac{ds}{\sqrt{f(s)}}.$$

The detailed consideration can be found in books and articles…

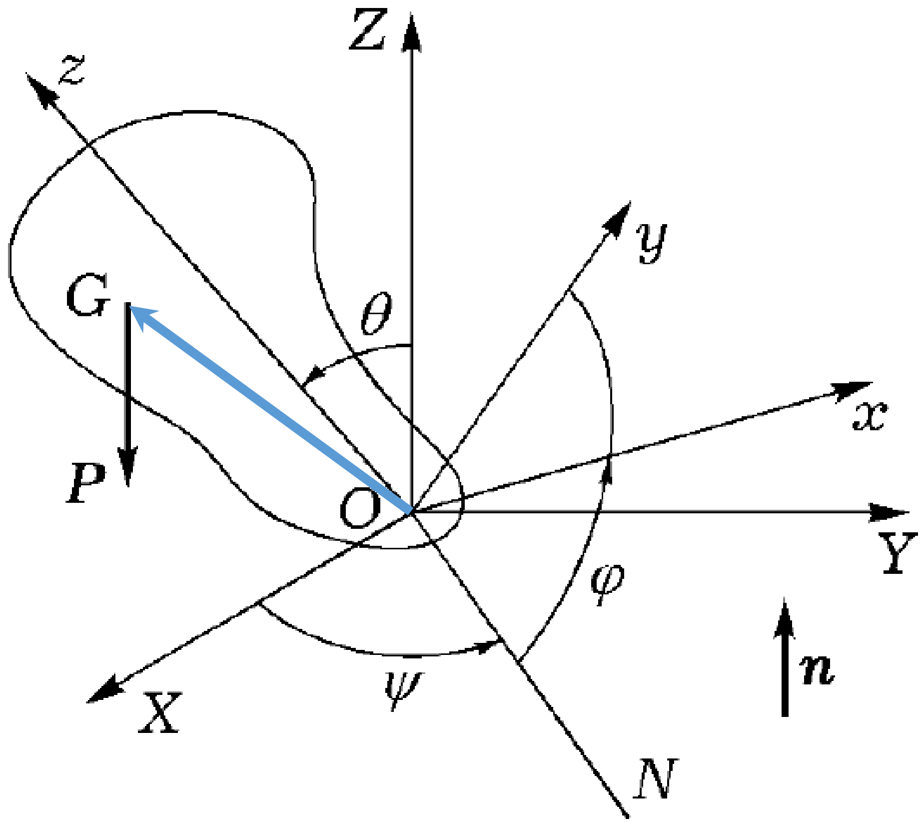
A real motion area



$$s_1 \leqslant s \leqslant s_2,$$

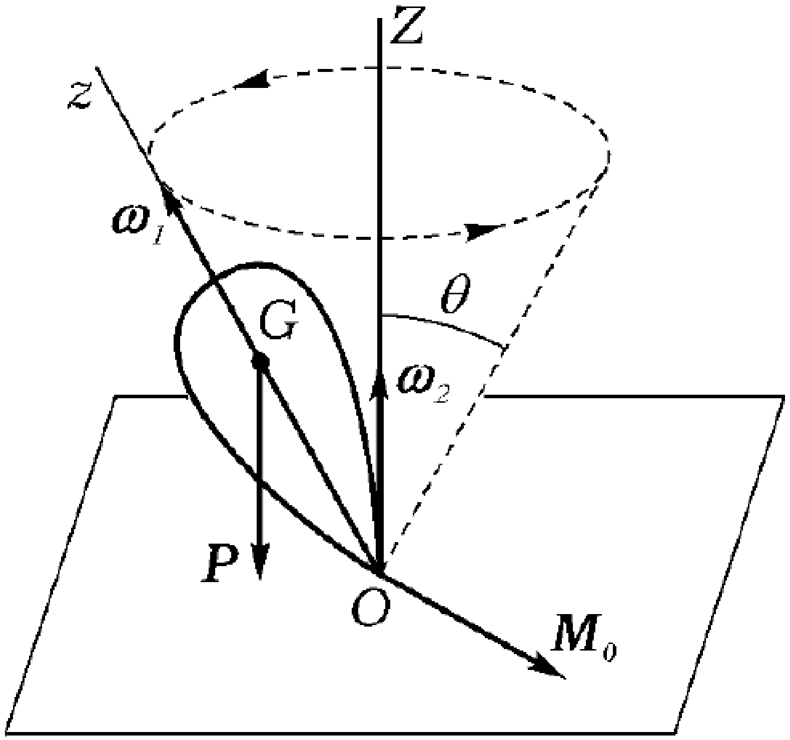
$$\cos\theta_1 \leqslant \cos\theta \leqslant \cos\theta_2,$$

Elementary gyroscopic effect



$$A = B,$$

$$A\frac{dp}{dt} + (C - A)qr = M_x,$$
$$A\frac{dq}{dt} - (C - A)rp = M_y,$$
$$C\frac{dr}{dt} = M_z. \qquad \text{(DYN.SYS)}$$



Let us to find conditions of the motion with **constant velocities of precession (ψ) and intrinsic rotation (φ) at constant nutation**:

$$\theta = \theta_0,\quad \dot\varphi = \omega_1,\quad \dot\psi = \omega_2$$

$$p = \dot\psi\sin\theta\sin\varphi + \dot\theta\cos\varphi,$$
$$q = \dot\psi\sin\theta\cos\varphi - \dot\theta\sin\varphi,$$
$$r = \dot\psi\cos\theta + \dot\varphi$$

$$p = \omega_2\sin\theta_0\sin\varphi,$$
$$q = \omega_2\sin\theta_0\cos\varphi,$$
$$r = \omega_2\cos\theta_0 + \omega_1.$$

(DYN.SYS)

$$M_x = \omega_2\omega_1\sin\theta_0\cos\varphi\left[C + (C - A)\frac{\omega_2}{\omega_1}\cos\theta_0\right]$$

$$M_y = -\omega_2\omega_1\sin\theta_0\sin\varphi\left[C + (C - A)\frac{\omega_2}{\omega_1}\cos\theta_0\right]$$

The main formula of the gyroscopic effect

$$M_x = \omega_2\omega_1 \sin\theta_0 \cos\varphi \left[C + (C - A)\frac{\omega_2}{\omega_1}\cos\theta_0\right]$$

$$M_y = -\omega_2\omega_1 \sin\theta_0 \sin\varphi \left[C + (C - A)\frac{\omega_2}{\omega_1}\cos\theta_0\right]$$

$$M_z = 0.$$

$$\theta = \theta_0$$
$$\dot{\varphi} = \omega_1$$
$$\dot{\psi} = \omega_2$$

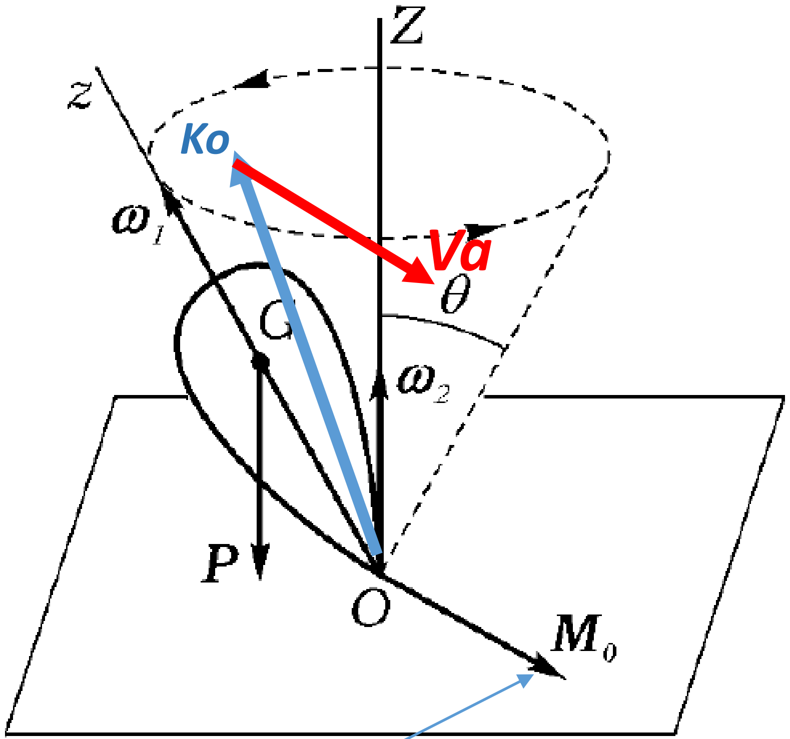
This can be reduced to the form:

$$\boldsymbol{M}_O = \boldsymbol{\omega}_2 \times \boldsymbol{\omega}_1 \left[C + (C - A)\frac{\omega_2}{\omega_1}\cos\theta_0\right]$$

Vector-part

Scalar-part

$$Va = \frac{d\boldsymbol{K}_O}{dt} = \boldsymbol{M}_O$$



The velocity ***Va*** of the end of the ***Ko*** vector is

1) equal to ***Mo*** value
2) directed along ***Mo*** vector

Let us now consider

$$\omega_1 \gg \omega_2$$

The next slide

The main formula of the gyroscopic effect

$$\boldsymbol{M}_O = \boldsymbol{\omega}_2 \times \boldsymbol{\omega}_1 \left[C + (C - A)\frac{\omega_2}{\omega_1}\cos\theta_0 \right]$$

$\theta = \theta_0$

$\dot{\varphi} = \omega_1$

$\dot{\psi} = \omega_2$

Let us now consider

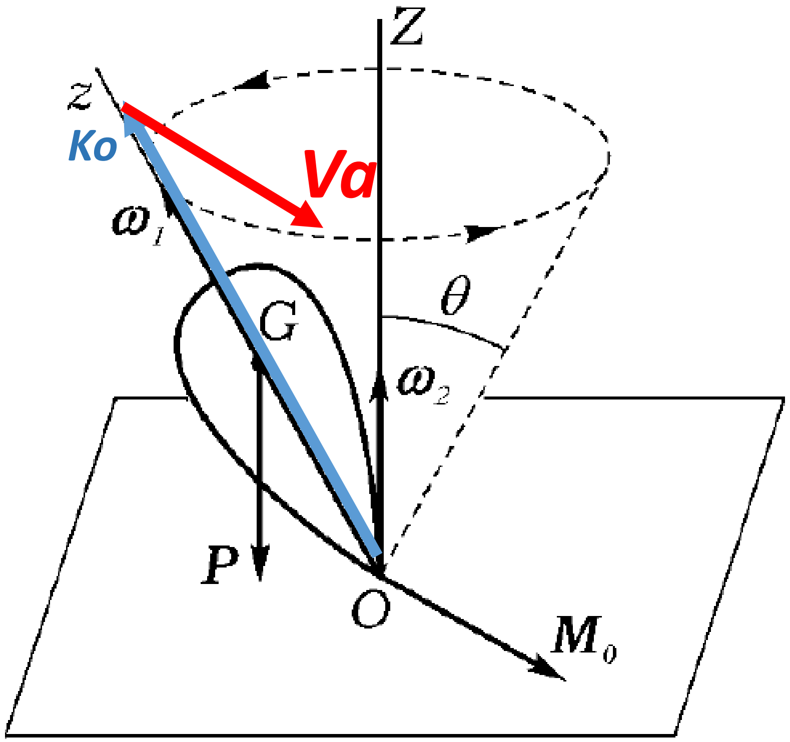
$\omega_1 \gg \omega_2$

$\frac{\omega_2}{\omega_1} \to 0$

$$\boldsymbol{M}_O = C\boldsymbol{\omega}_2 \times \boldsymbol{\omega}_1.$$

If assume

$\boldsymbol{K}_O = C\boldsymbol{\omega}_1.$

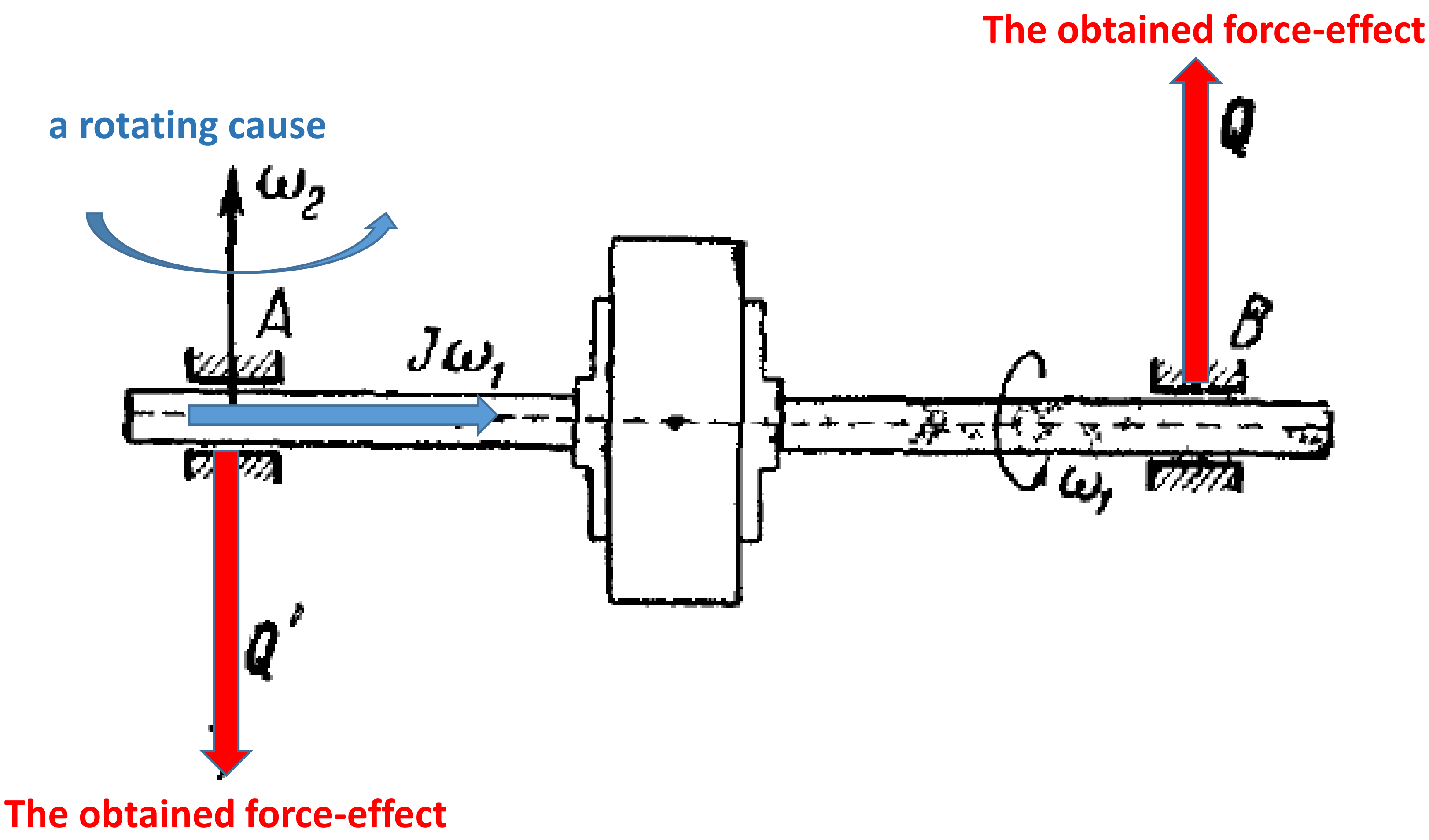


A backward consideration: if we intend to rotate the spun rigid body, the REACTION-torque will arise $\boldsymbol{M} = -\boldsymbol{M}_O.$

This is the gyroscopic torque $\boldsymbol{M} = C\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2.$

The main formula of the gyroscopic effect

The gyroscopic torque $\boldsymbol{M} = C\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2.$



Sofia Vasilyevna Kovalevskaya

Born: 15 January 1850
Moscow, Russian Empire
Died: 10 February 1891
Stockholm, Sweden

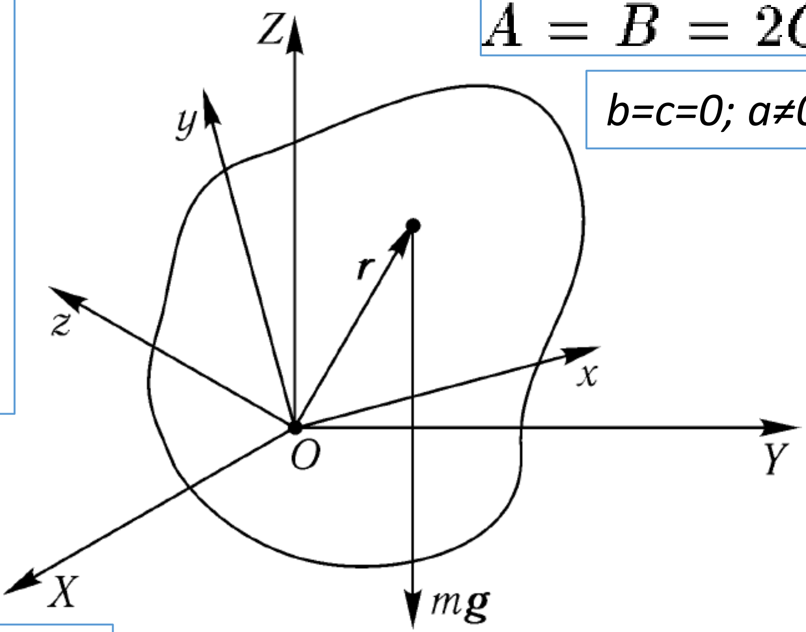
S.V. KOVALEVSKAYA'S CASE OF RIGID BODY MOTION

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3, \quad \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1, \quad \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2.$$

$$A\frac{dp}{dt} + (C - B)qr = P(\gamma_2 c - \gamma_3 b),$$
$$B\frac{dq}{dt} + (A - C)rp = P(\gamma_3 a - \gamma_1 c),$$
$$C\frac{dr}{dt} + (B - A)pq = P(\gamma_1 b - \gamma_2 a).$$

$$A = B = 2C$$

b=c=0; a≠0



The first integrals:

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

$$\boldsymbol{K}_O \cdot \boldsymbol{n} = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = \text{const}.$$

$$\frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + P(a\gamma_1 + b\gamma_2 + c\gamma_3) = \text{const}.$$

$$(p^2 - q^2 - \alpha\gamma_1)^2 + (2pq - \alpha\gamma_2)^2 = \text{const}.$$

where $\left(\alpha = \frac{Pa}{C}\right)$

THE S.V. KOVALEVSKAYA TOP

In 1888 she won the ***Prix Bordin* of the French Academy of Science**, for her work on the question: *"Mémoire sur un cas particulier du problème de le rotation d'un corps pesant autour d'un point fixe, où l'intégration s'effectue à l'aide des fonctions ultraelliptiques du temps"*

$$a_1 = a_2 = 1,\ a_3 = 2,\ r_2 = r_3 = 0,\ r_1 = x, \quad \mu = mg$$

$$\mathbf{I} = \mathrm{diag}(I_1, I_2, I_3) \qquad \mathbf{A} = \mathbf{I}^{-1} \qquad \boldsymbol{M} = \mathbf{I}\boldsymbol{\omega}$$

$$\begin{cases} \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \mu \boldsymbol{r} \times \boldsymbol{\gamma}, \\ \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \end{cases}$$

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) - x\gamma_1$$

$$F_1 = (\boldsymbol{M}, \boldsymbol{\gamma}), \quad F_2 = \boldsymbol{\gamma}^2.$$

Kovalevskaya's first intrgral:

$$F_3 = \left(\frac{M_1^2 - M_2^2}{2} + x\gamma_1\right)^2 + (M_1 M_2 + x\gamma_2)^2 = k^2$$

THE S.V. KOVALEVSKAYA TOP

Kovalevskaya's variables:

$$s_1 = \frac{R - \sqrt{R_1 R_2}}{2(z_1 - z_2)^2}, \qquad s_2 = \frac{R + \sqrt{R_1 R_2}}{2(z_1 - z_2)^2},$$

$$z_1 = M_1 + iM_2, \qquad z_2 = M_1 - iM_2,$$

$$R = R(z_1,\, z_2) = \frac{1}{4} z_1^2 z_2^2 - \frac{h}{2}(z_1^2 + z_2^2) + c(z_1 + z_2) + \frac{k^2}{4} - 1,$$

$$R_1 = R(z_1,\, z_1), \qquad R_2 = R(z_2,\, z_2),$$

Where: $F_1 = (M,\, \gamma) = c,\; H = h.$

The motion equations in Kovalevskaya's variables:

$$\frac{ds_1}{\sqrt{P(s_1)}} = \frac{dt}{s_1 - s_2}, \qquad \frac{ds_2}{\sqrt{P(s_2)}} = \frac{dt}{s_2 - s_1},$$

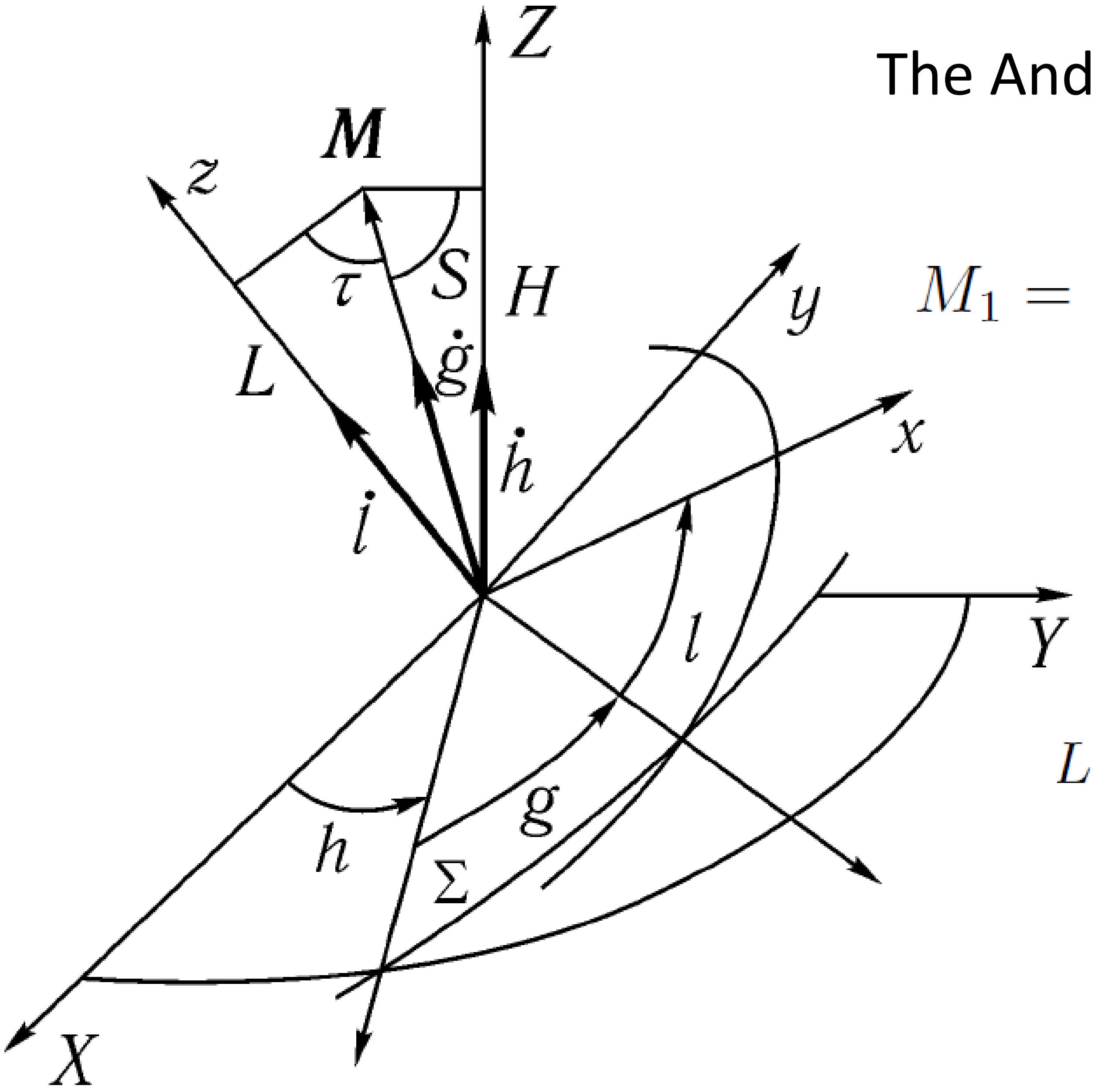
Polynomial of 5$^{\text{th}}$ power

$$P(s) = \left(\left(2s + \frac{h}{2}\right)^2 - \frac{k^2}{16}\right)\left(4s^3 + 2hs^2 + \left(\frac{h^2}{4} - \frac{k^2}{16} + \frac{1}{4}\right)s + \frac{c^2}{16}\right).$$

Then the hyperelliptic integrals/functions follow…

The end of classical cases of the rigid body motion…

Special aspects of the rigid body dynamics



The Andoyer–Deprit variables

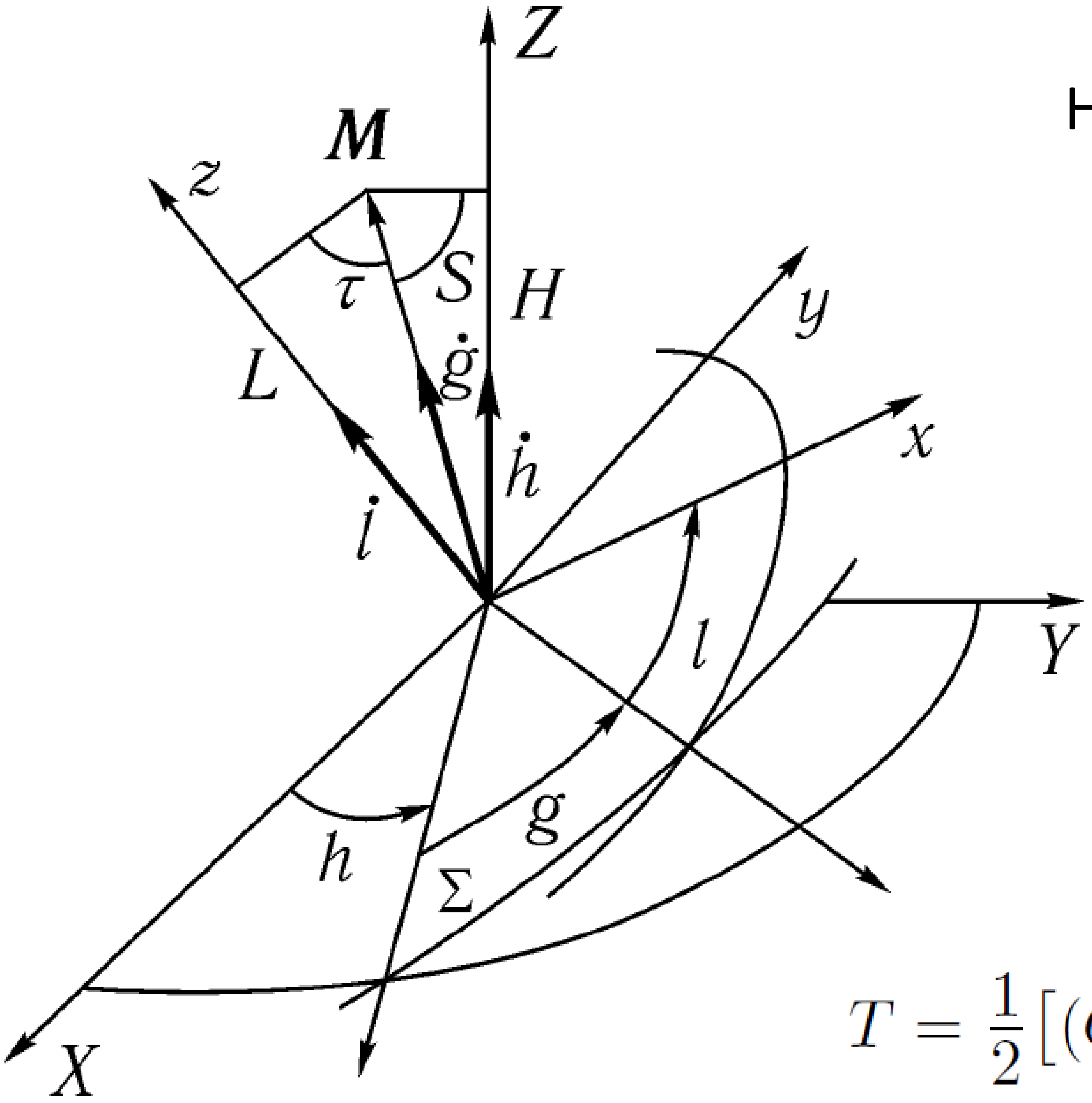
$$M_1 = \sqrt{G^2 - L^2}\sin l, \quad M_2 = \sqrt{G^2 - L^2}\cos l,$$

$$M_3 = L, \quad G^2 = \boldsymbol{M}^2$$

$$L = M_3, G = \sqrt{(\boldsymbol{M}, \boldsymbol{M})}, l = \operatorname{arctg}\left(\frac{M_1}{M_2}\right),$$

$$g = \arcsin\left(\frac{M_2\gamma_1 - M_1\gamma_2}{\sqrt{M_1^2 + M_2^2}}\right).$$

Special aspects of the rigid body dynamics



Hamiltonian form of equations:

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}, \quad \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}},$$

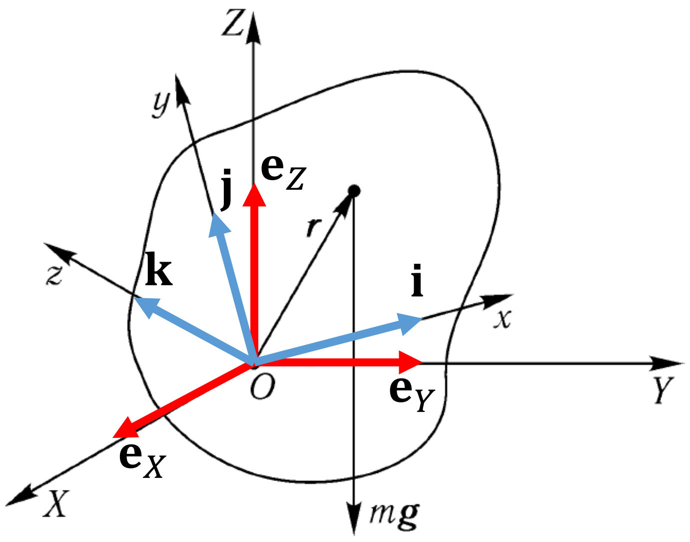
$$H = T + U$$

The potential and kinetic energy in the A-D variables:

$$U = U(L,\, G,\, H,\, l,\, g,\, h).$$

$$T = \frac{1}{2}\left[(G^2 - L^2)(a_1 \sin^2 l + a_2 \cos^2 l) + a_3 L^2\right].$$

Directional cosines can be expressed in A-D variables:



$$\begin{cases} \mathbf{e}_X = \alpha_1 \mathbf{i} + \alpha_2 \mathbf{j} + \alpha_3 \mathbf{k} \\ \mathbf{e}_Y = \beta_1 \mathbf{i} + \beta_2 \mathbf{j} + \beta_3 \mathbf{k} \\ \mathbf{e}_Z = \gamma_1 \mathbf{i} + \gamma_2 \mathbf{j} + \gamma_3 \mathbf{k} \end{cases}$$

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix}$$

$$\alpha_1 = -\sin l \sin h \cos g \sin \tau \sin \zeta + \sin l \sin h \cos \tau \cos \zeta - \\ - \sin l \sin g \cos h \sin \tau - \cos l \sin h \sin g \sin \zeta + \cos l \cos g \cos h,$$

$$\alpha_2 = \cos l \cos g \sin h \sin \tau \sin \zeta - \cos l \sin h \cos \tau \cos \zeta + \\ + \cos l \cos h \sin g \sin \tau - \sin l \sin g \sin \zeta \sin h + \sin l \cos h \cos g,$$

$$\alpha_3 = \sin h \cos \tau \cos g \sin \zeta + \sin h \sin \tau \cos \zeta + \cos \tau \sin g \cos h,$$

$$\beta_1 = -(\sin l \cos h \cos g \sin \tau \sin \zeta - \sin l \cos h \cos \zeta \cos \tau - \\ - \sin l \sin g \sin h \sin \tau + \cos l \cos h \sin g \sin \zeta + \cos l \cos g \sin h),$$

$$\beta_2 = \cos l \cos h \sin \tau \cos g \sin \zeta - \cos l \cos h \cos \zeta \cos \tau - \\ - \cos l \sin g \sin h \sin \tau - \sin l \cos h \sin g \sin \zeta - \sin l \cos g \sin h,$$

$$\beta_3 = -\sin h \cos \tau \sin g + \cos \tau \cos g \sin \zeta \cos h + \sin \tau \cos \zeta \cos h,$$

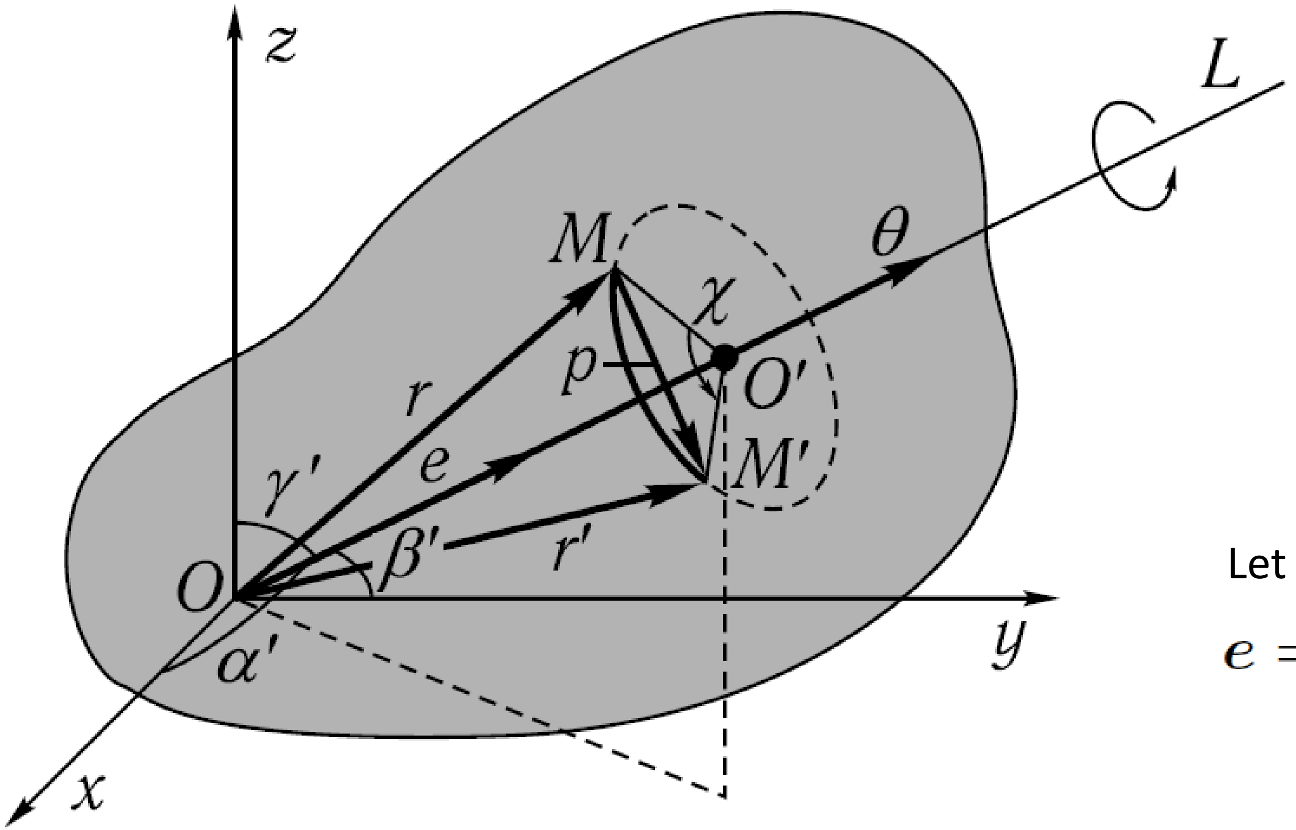
$$\gamma_1 = (\sin \zeta \cos \tau + \sin \tau \cos \zeta \cos g) \sin l + \cos \zeta \sin g \cos l,$$

$$\gamma_2 = (\sin \zeta \cos \tau + \sin \tau \cos \zeta \cos g) \cos l - \cos \zeta \sin g \sin l,$$

$$\gamma_3 = \sin \zeta \sin \tau - \cos \tau \cos \zeta \cos g,$$

$$\sin \tau = \frac{L}{G}, \ \sin \zeta = \frac{H}{G}.$$

The Euler parameters / Euler–Rodrigues formula



$$\boldsymbol{p} = \overrightarrow{OM}' - \overrightarrow{OM} = \boldsymbol{r}' - \boldsymbol{r}$$

$$\boldsymbol{p} = \frac{1}{1 + \frac{1}{4}\theta^2}\boldsymbol{\theta} \times \left(\boldsymbol{r} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{r}\right),$$

$$\boldsymbol{\theta} = 2\,\mathrm{tg}\,\frac{\chi}{2}\boldsymbol{e}$$

Let we have the unit-vector

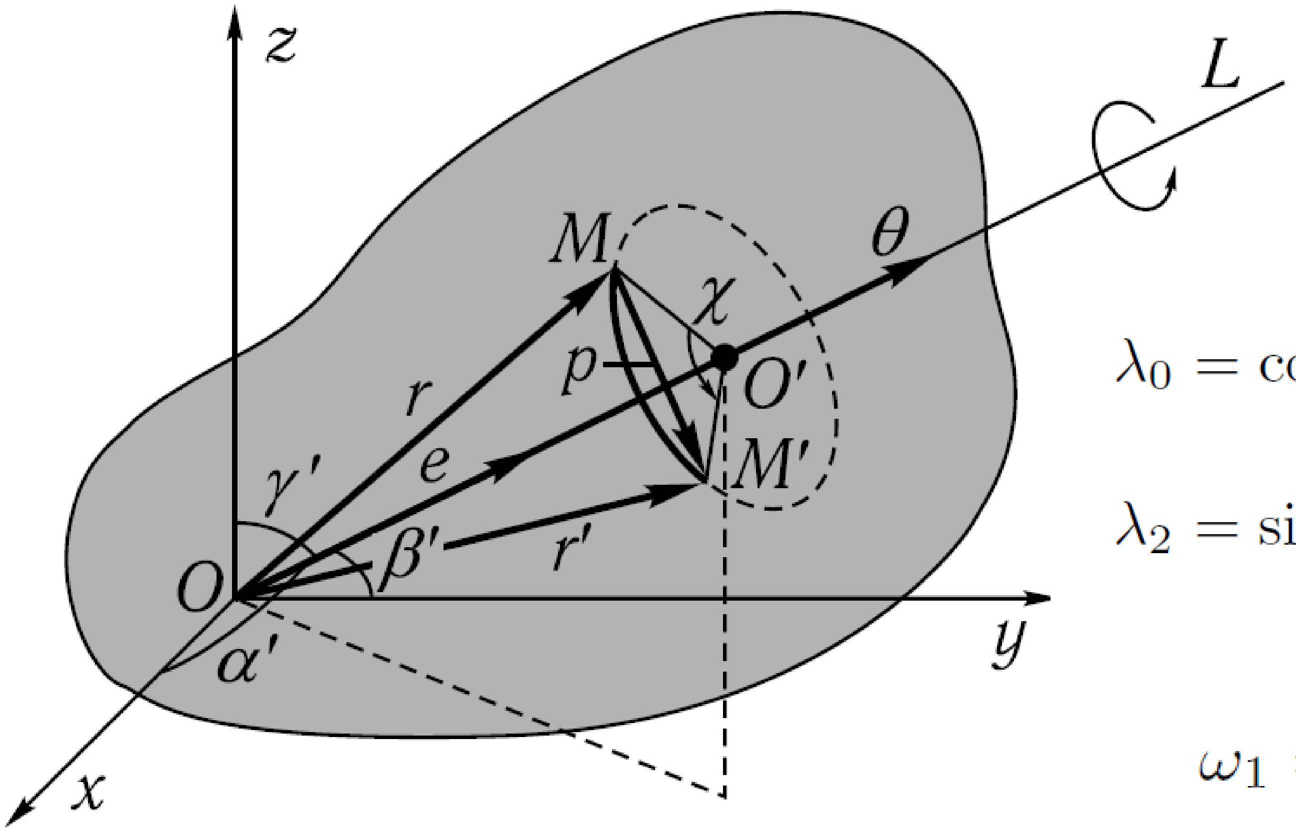
$$\boldsymbol{e} = \boldsymbol{i}\cos\alpha' + \boldsymbol{j}\cos\beta' + \boldsymbol{k}\cos\gamma',$$

Then the E-R parameters are:

$$\lambda_0 = \cos\frac{\chi}{2}, \qquad \lambda_1 = \cos\alpha'\sin\frac{\chi}{2},$$

$$\lambda_2 = \cos\beta'\sin\frac{\chi}{2}, \qquad \lambda_3 = \cos\gamma'\sin\frac{\chi}{2}$$

The Euler parameters / Euler–Rodrigues formula



Connections with the Euler angles and the angular velocity components:

$$\lambda_0 = \cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}, \qquad \lambda_1 = \sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2},$$

$$\lambda_2 = \sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}, \qquad \lambda_3 = \cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}.$$

$$\omega_1 = 2(\lambda_0\dot\lambda_1 + \lambda_3\dot\lambda_2 - \lambda_2\dot\lambda_3 - \lambda_1\dot\lambda_0),$$

$$\omega_2 = 2(-\lambda_3\dot\lambda_1 + \lambda_0\dot\lambda_2 + \lambda_1\dot\lambda_3 - \lambda_2\dot\lambda_0),$$

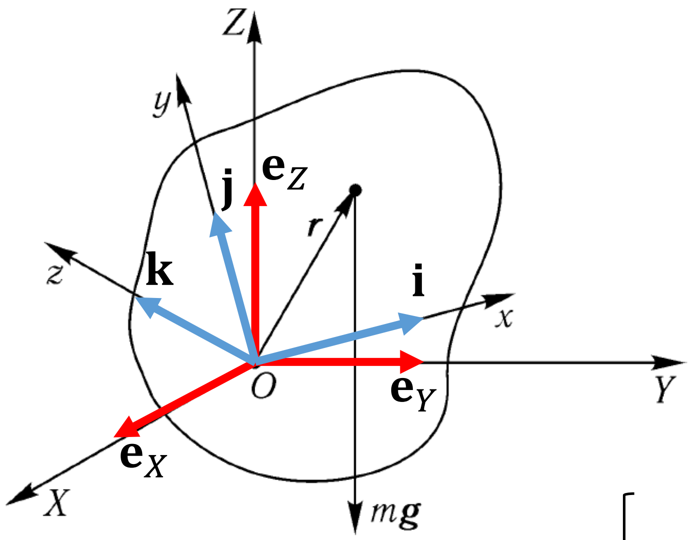
$$\omega_3 = 2(\lambda_2\dot\lambda_1 - \lambda_1\dot\lambda_2 + \lambda_0\dot\lambda_3 - \lambda_3\dot\lambda_0).$$

The Euler parameters / Euler–Rodrigues formula

Connections with the directional cosines:

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\psi - \cos\theta\sin\psi\sin\varphi & \cos\varphi\sin\psi + \cos\theta\cos\psi\sin\varphi & \sin\varphi\sin\theta \\ -\sin\varphi\cos\psi - \cos\theta\sin\psi\cos\varphi & -\sin\varphi\sin\psi + \cos\theta\cos\psi\cos\varphi & \cos\varphi\sin\theta \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{pmatrix}.$$

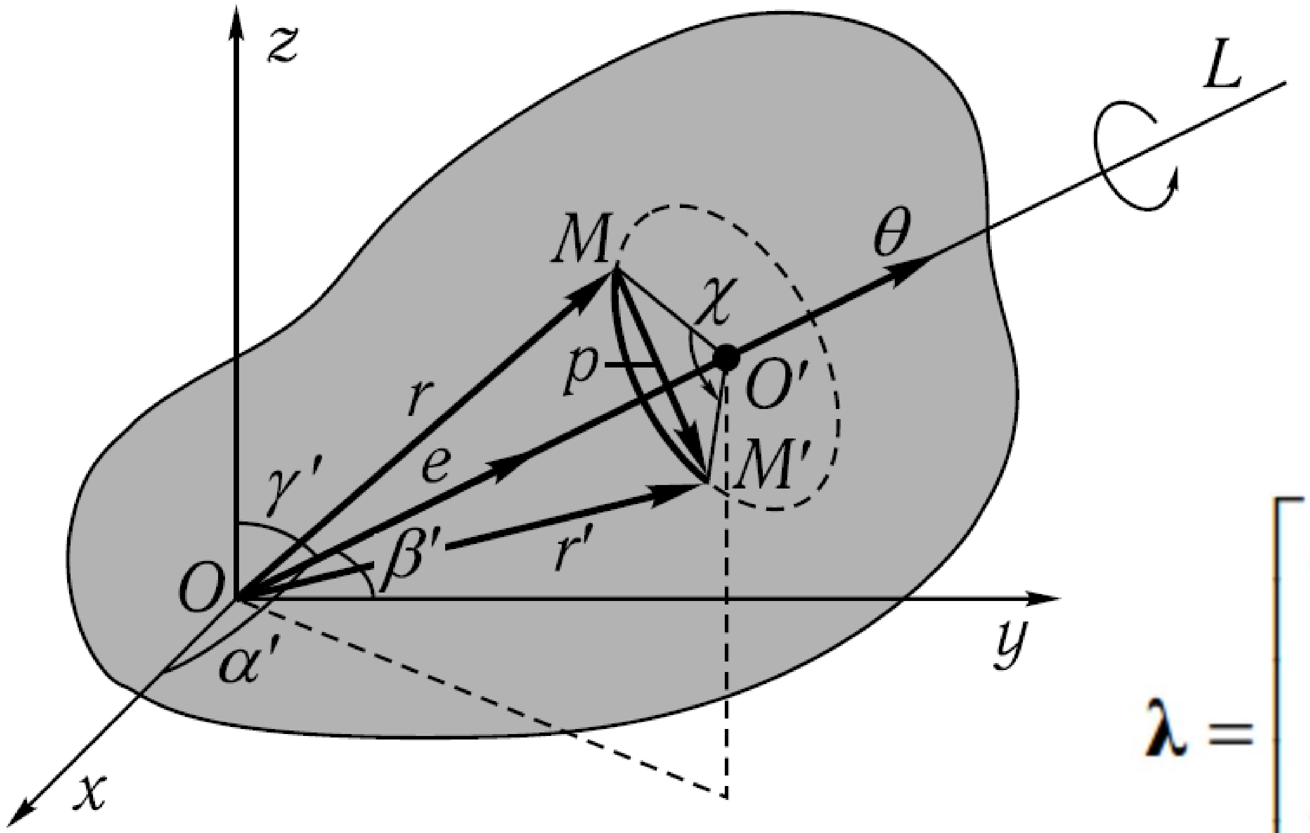
$$= \begin{pmatrix} \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2 & 2(\lambda_0\lambda_3 + \lambda_1\lambda_2) & 2(\lambda_1\lambda_3 - \lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2 - \lambda_0\lambda_3) & \lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2 & 2(\lambda_0\lambda_1 + \lambda_2\lambda_3) \\ 2(\lambda_0\lambda_2 + \lambda_1\lambda_3) & 2(\lambda_2\lambda_3 - \lambda_0\lambda_1) & \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2 \end{pmatrix}.$$



Then we have the connections of the angles and parameters… - we can use it for rewriting of torques expressions and building of dynamical equations… :

$$\begin{cases} A\dfrac{dp}{dt} + (C - B)qr = P(\gamma_2 c - \gamma_3 b) = P\left(2\left(\lambda_0\lambda_1 + \lambda_2\lambda_3\right)c - \left(\lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2\right)b\right) \\ B\ldots \\ C\ldots \end{cases}$$

The Euler parameters / Euler–Rodrigues formula



$$\lambda_0^2 + \boldsymbol{\lambda}^2 = 1, \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$$

Kinematical equations:

$$2\dot{\boldsymbol{\lambda}} = \boldsymbol{\Theta} \cdot \boldsymbol{\lambda}$$

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}, \quad \boldsymbol{\Theta} = \begin{bmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{bmatrix}$$